Question J-01 [4 points]

The 2020 students in a school are divided into 7 groups to compete in a game. Any two groups can have the same number of students. What is the minimum number of students in the largest group(s)?

一间学校有 2020 位学生。如果将这些学生分成 7 组来进行运动会，任意两组可以有相同的人数，那么人数最多的组至少有几个人?

Answer: [289]

Solutions:

$$2020 = 288 \times 7 + 4.$$

If each group has at most 288 students, then there are at most 2016 students, a contradiction. It can be that 4 groups have 289 students, while 3 groups have 288. Therefore, the largest group(s) has at least 289 students.

Question J-02 [4 points]

There are 11 students that took part in a mathematics exam. By taking 10 students at a time and sum up their marks, the following 11 sums are obtained:

$$749 \quad 743 \quad 740 \quad 737 \quad 733 \quad 729 \quad 726 \quad 723 \quad 718 \quad 714 \quad 708$$

What is the lowest mark scored by these 11 students?

有 11 位学生参加数学考试。每次取 10 位学生的分数加起来，得到以下的 11 个和:

$$749 \quad 743 \quad 740 \quad 737 \quad 733 \quad 729 \quad 726 \quad 723 \quad 718 \quad 714 \quad 708$$

这 11 位学生分数最低的是几分?

Answer: [53]

Solutions:

The sum of the 11 sums is 8020, which is 10 times the sum of the marks of the 11 students. Hence, the lowest mark scored by the students is $802 - 749 = 53$.

Question J-03 [4 points]

Tianying Company has N employees, $\frac{3}{8}$ of them work in the Johor branch, and 15% of them work in the Penang branch. The remaining 437 employees work in the headquarter in Kuala Lumpur. Find N.

天英公司有 N 位员工，$\frac{3}{8}$ 的员工在柔佛分部，15% 的员工在槟城分部，剩下的 437 位员工在吉隆坡总部。求 N 。

Answer: [920]

Solutions:

$$N - \frac{3}{8}N - \frac{3}{20}N = 437$$
$$N = 920$$

Question J-04 [4 points]

If m and n are positive integers such that $n^5 = 18000m$, find the smallest possible value of m.

已知 m 与 n 是正整数且 $n^5 = 18000m$，求 m 的最小可能值。

Answer: [1350]

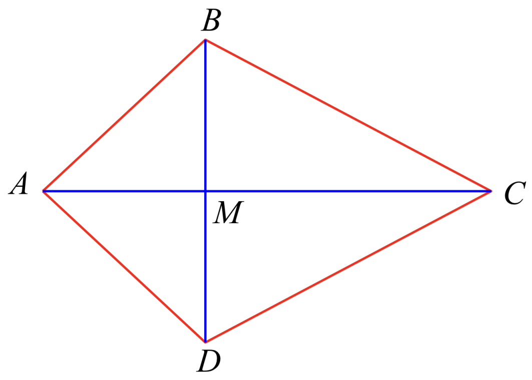
Solutions:

$18000 = 2^4 \times 3^2 \times 5^3$. The smallest possible value of m is $2 \times 3^3 \times 5^2 = 1350$.

Question J-05 [4 points]

In the figure shown below, $ABCD$ is a kite. Given that $AM = 15$, $CM = 48$, and the area of the kite is 1260, find the perimeter of the kite.

下图中， $ABCD$ 是风筝形。已知 $AM = 15$， $CM = 48$，风筝形的面积为 1260。求风筝形的周长。



Answer: [154]

Solutions:

$$BD \times (48 + 15) = 1260 \times 2$$
$$BM = \frac{BD}{2} = 20$$

Therefore, $AD = 25$, $CD = 52$. The perimeter of the kite is $2 \times (25 + 52) = 154$.

Question J-06 [4 points]

There are two balls, one red and one yellow. The volume of the yellow ball is 72.8% more than the volume of the red ball, while the surface area is $x\%$ more. Find x.

有两粒球，一粒红色，一粒黄色。黄球的体积比红球的大 72.8%，表面积比红球的大 $x\%$，求 x 。

Answer: [44]

Solutions:

Let the radii of the red ball and the yellow ball be r_1 and r_2 respectively. Then

$$\left(\frac{r_2}{r_1}\right)^3 = 1.728.$$

Hence, $r_2 = 1.2r_1$.

The ratio of their surface areas is

$$\left(\frac{r_2}{r_1}\right)^2 = 1.44.$$

Hence, $x = 44$.

Question J-07 [4 points]

Given that x and y are real numbers such that $\dfrac{11y-6x}{17x+8y}=\dfrac{294}{577}$, find $\dfrac{6x+11y}{17x-8y}$.

已知 x 与 y 是实数且 $\dfrac{11y-6x}{17x+8y}=\dfrac{294}{577}$，求 $\dfrac{6x+11y}{17x-8y}$。

Answer: [498]

Solutions:

Let $k=\dfrac{x}{y}$. Then

$$\frac{294}{577}=\frac{11-6k}{17k+8}$$

This gives $k=\dfrac{17}{36}$. Hence,

$$\frac{6x+11y}{17x-8y}=\frac{6k+11}{17k-8}=498.$$

Question J-08 [4 points]

Given that all the two-digit positive integers are divided into 5 groups such that the sum of the numbers in each group is N. Find N.

将所有正的二位数分成 5 组，使得每组的和都是 N。求 N。

Answer: [981]

Solutions:

The two-digit positive integers are 10, 11, 12, $\ldots$, 99. There are 90 of them. Their sum is

$$\frac{10+99}{2}\times 90.$$

Hence,

$$N=\frac{10+99}{2}\times\frac{90}{5}=981.$$

Question J-09 [4 points]

If a and b are real numbers such that $a-23 \neq 0$, $3b+37 \neq 0$, $a \neq 3b+60$ and

$$\frac{2}{a-23} - \frac{2}{3b+37} = 3b - a + 60,$$

find the value of $3ab + 37a - 69b$.

已知 a 与 b 是实数。$a-23 \neq 0$，$3b+37 \neq 0$，$a \neq 3b+60$ 且

$$\frac{2}{a-23} - \frac{2}{3b+37} = 3b - a + 60$$

求 $3ab + 37a - 69b$ 的值。

Answer: [853]

Solutions:

Let $u = a - 23$, $v = 3b + 37$. Then $u - v = a - 3b - 60 \neq 0$.

$$\frac{2}{u} - \frac{2}{v} = -(u - v)$$
$$\frac{2(u-v)}{uv} - (u - v) = 0$$
$$(u - v)\left(\frac{2}{uv} - 1\right) = 0$$

Since $u - v \neq 0$, we have $uv = 2$. This gives

$$(a - 23)(3b + 37) = 2$$
$$3ab + 37a - 69b = 853$$

Question J-10 [4 points]

How many four-digit numbers leaves a remainder of 7 when divided by 13?

有所少个四位数除以 13 时余数为 7 ?

Answer: [692]

Solutions:

$$1008 = 77 \times 13 + 7$$

$$9991 = 768 \times 13 + 7$$

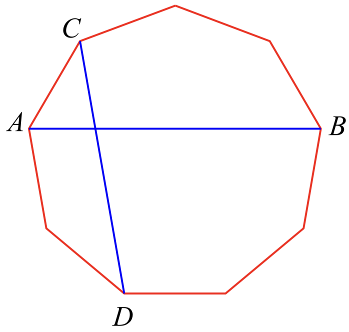
The number of four-digit number that leaves a remainder of 7 when divided by 13 is

$$768 - 76 = 692.$$

Question J-11 [5 points]

The figure shown below is a regular 9-gon. If the acute angle between the diagonals AB and CD is $x°$, find x.

下图所示为一正九边形。若对角线 AB 与 CD 所夹的锐角为 $x°$，求 x。



Answer: [80]

Solutions:

Each internal angle of the regular 9-gon is $\frac{7 \times 180°}{9} = 140°$. Hence, $\angle ACD = 40°$.

If $\angle CAB = y°$, then considering the sum of angles in ΔABC, we have

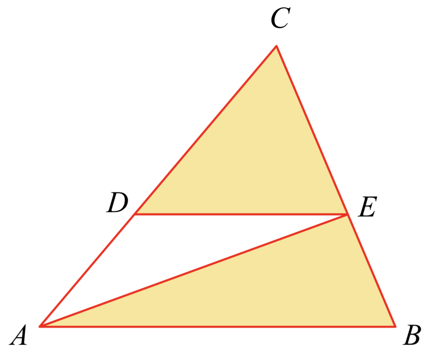
$$y + 140 - 40 + y - 40 = 180.$$

This gives $y = 60$ and hence $x = 80$.

Question J-12 [5 points]

In the figure shown below, $DE//AB$, $AD : DC = m : n$, where m and n are relatively prime positive integers and $m < n$. If the area of the shaded region is $\frac{403}{529}$ of the area of the whole triangle ΔABC, find the value of n.

下图中， $DE//AB$ ， $AD : DC = m : n$，其中 m 及 n 是互质的正整数且 $m < n$。如果着色部分的面积占整个三角形 ΔABC 面积的 $\frac{403}{529}$，求 n 的值。



Answer: [14]

Solutions:

$$\frac{S_{\Delta CDE}}{S_{\Delta ABC}} = \left(\frac{n}{m+n}\right)^2$$
$$\frac{S_{\Delta ABE}}{S_{\Delta ABC}} = \frac{m}{m+n}$$

Let $k = \frac{n}{m}$. Then

$$\frac{k^2}{(1+k)^2} + \frac{1}{1+k} = \frac{403}{529}$$
$$\frac{k^2+k+1}{k^2+2k+1} = \frac{403}{529}$$
$$\frac{k}{k^2+2k+1} = \frac{126}{529}$$
$$k+2+\frac{1}{k} = \frac{529}{126}$$
$$k^2 - \frac{277}{126}k + 1 = 0$$
$$\left(k - \frac{14}{9}\right)\left(k - \frac{9}{14}\right) = 0$$

Since $k > 1$, we have $k = \frac{14}{9}$ and $n = 14$.

Question J-13 [5 points]

How many pairs of positive integers (x, y) satisfy the equation $101x + 20y = 1010101$?

有多少对正整数 (x, y) 满足方程式 $101x + 20y = 1010101$?

Answer: [500]

Solutions:

Observe that $101 - 20 \times 5 = 1$. Hence,

$$101 \times 1010101 - 20 \times 5050505 = 1010101.$$

If $101x + 20y = 1010101$, then

$$101(x - 1010101) + 20(y + 5050505) = 0.$$

Since 20 and 101 are relatively prime, there must exist integer k such that

$$x - 1010101 = -20k$$

$$y + 5050505 = 101k$$

To have $x > 0$ and $y > 0$, we must have $50005 < k \leq 50505$.

There are 500 pairs of such (x, y).

Question J-14 [5 points]

Given that

$$\frac{xy}{11x+12y}=\frac{110}{21}, \qquad \frac{xy}{10x+11y}=\frac{132}{23}.$$

Find the value of $x+y$.

已知

$$\frac{xy}{11x+12y}=\frac{110}{21}, \qquad \frac{xy}{10x+11y}=\frac{132}{23}.$$

求 $x+y$ 的值.

Answer: [242]

Solutions:

$$\frac{11}{y}+\frac{12}{x}=\frac{21}{110}$$
$$\frac{10}{y}+\frac{11}{x}=\frac{23}{132}.$$

From these, we find that $x=132$, $y=110$ and hence $x+y=242$.

Question J-15 [5 points]

Given that

$(1+x)+(1+2x)^2+(1+3x)^3+(1+4x)^4+(1+5x)^5 = a_0+a_1x+a_2x^2+\ldots+a_{25}x^{25}$.

Find $a_2+a_4+a_6+\ldots+a_{24}$.

已知

$(1+x)+(1+2x)^2+(1+3x)^3+(1+4x)^4+(1+5x)^5 = a_0+a_1x+a_2x^2+\ldots+a_{25}x^{25}$

求 $a_2+a_4+a_6+\ldots+a_{24}$。

Answer: [3758]

Solutions:

Put $x=0$, $x=1$ and $x=-1$, we obtain respectively

$$a_0 = 5$$
$$a_0+a_1+\ldots+a_{24}+a_{25} = 2+3^2+4^3+5^4+6^5$$
$$a_0-a_1+a_2+\ldots+a_{24}-a_{25} = (-1)^2+(-2)^3+(-3)^4+(-4)^5$$

The sum of the second and third equations divided by two, and subtract the first equation give

$$a_2+a_4+a_6+\ldots+a_{24} = 3758.$$

Question J-16 [5 points]

A school has four clubs whose members are students in this school. Each club has 99 members. Every two clubs have 33 common members. Every three clubs have 11 common members. There is exactly one student that joins all four clubs. At least how many students does this school have?

已知一间学校有 4 个学会，每个学会的会员都是该校的学生。每个学会有 99 位会员，每两个学会有 33 位共同会员，每三个学会有 11 位共同会员，恰有一位学生是这四个学会的共同会员。这间学校最少有几位学生?

Answer: [241]

Solutions:

Let A_1, A_2, A_3, A_4 be respectively the set of students in the club 1, club 2, club 3 and club 4.

$$\begin{aligned} n(A_1 \cup A_2 \cup A_3 \cup A_4) =& n(A_1) + n(A_2) + n(A_3) + n(A_4) - n(A_1 \cap A_2) - n(A_1 \cap A_3) \\ & - n(A_1 \cap A_4) - n(A_2 \cap A_3) - n(A_2 \cap A_4) - n(A_3 \cap A_4) \\ & + n(A_1 \cap A_2 \cap A_3) + n(A_1 \cap A_2 \cap A_4) + n(A_1 \cap A_3 \cap A_4) \\ & + n(A_2 \cap A_3 \cap A_4) - n(A_1 \cap A_2 \cap A_3 \cap A_4) \\ =& 99 \times 4 - 33 \times 6 + 11 \times 4 - 1 \\ =& 241 \end{aligned}$$

Question J-17 [5 points]

Find the sum of the real solutions to the equation

$$\frac{1}{x-12}+\frac{6}{x-14}=\frac{3}{x-17}+\frac{4}{x-11}.$$

求方程式

$$\frac{1}{x-12}+\frac{6}{x-14}=\frac{3}{x-17}+\frac{4}{x-11}$$

的所有实数解的和。

Answer: [85]

Solutions:

$$\begin{aligned}&\frac{1}{x-12}+\frac{6}{x-14}-\frac{3}{x-17}-\frac{4}{x-11}\\=&\frac{x^2-85x+886}{(x-11)(x-12)(x-14)(x-17)}\end{aligned}$$

The solutions of the equation are solutions of

$$x^2-85x+886=0.$$

Since $\Delta = 85^2 - 4\times 886 > 0$, there are two real solutions whose sum is 85.

Question J-18 [5 points]

Let

$$a_n = \left\lfloor \frac{n}{13} \right\rfloor,$$

where $\lfloor x \rfloor$ is the largest integer not larger than x. Find the sum

$$a_1 + a_2 + \ldots + a_{200}.$$

设

$$a_n = \left\lfloor \frac{n}{13} \right\rfloor$$

其中 $\lfloor x \rfloor$ 是不大于 x 的最大整数。求

$$a_1 + a_2 + \ldots + a_{200}$$

Answer: [1455]

Solutions:

$$\begin{aligned}
S =& a_1 + a_2 + \ldots + a_{200} \\
=& \left(\left\lfloor \frac{1}{13} \right\rfloor + \left\lfloor \frac{2}{13} \right\rfloor + \ldots + \left\lfloor \frac{12}{13} \right\rfloor \right) \\
&+ \left(\left\lfloor \frac{13}{13} \right\rfloor + \left\lfloor \frac{14}{13} \right\rfloor + \ldots + \left\lfloor \frac{25}{13} \right\rfloor \right) \\
&+ \left(\left\lfloor \frac{26}{13} \right\rfloor + \left\lfloor \frac{27}{13} \right\rfloor + \ldots + \left\lfloor \frac{38}{13} \right\rfloor \right) \\
&+ \qquad \qquad \vdots \\
&+ \left(\left\lfloor \frac{182}{13} \right\rfloor + \left\lfloor \frac{183}{13} \right\rfloor + \ldots + \left\lfloor \frac{194}{13} \right\rfloor \right) \\
&+ \left(\left\lfloor \frac{195}{13} \right\rfloor + \left\lfloor \frac{196}{13} \right\rfloor + \left\lfloor \frac{197}{13} \right\rfloor + \left\lfloor \frac{198}{13} \right\rfloor + \left\lfloor \frac{199}{13} \right\rfloor + \left\lfloor \frac{200}{13} \right\rfloor \right) \\
=& 13 \times (1 + 2 + \ldots + 14) + 15 \times 6 \\
=& 1455
\end{aligned}$$

Question J-19 [5 points]

Let

$$S = \left\{ r \mid r \text{ is the remainder when } n^2 \text{ is divided by } 100\text{, where } n \text{ is an integer} \right\}.$$

Find the sum of the elements in S.

设

$$S = \left\{ r \mid r \text{ 是 } n^2 \text{ 除以 } 100 \text{ 的余数，其中 } n \text{ 是整数} \right\}$$

求 S 中元素的和。

Answer: [975]

Solutions:

Write $n = 10k + u$, where k is an integer, and u is an integer between 0 and 9. Then

$$n^2 = 100k^2 + 20ku + u^2$$

We just need to consider k taking values 0, 1, 2, 3, 4.

When $u = 0$, we find that S contains 0.

When $u = 1, 9$, we find that S contains $1, 21, 41, 61, 81$.

When $u = 2, 8$, we find that S contains $4, 44, 84, 24, 64$.

When $u = 3, 7$, we find that S contains $9, 69, 29, 89, 49$.

When $u = 4, 6$, we find that S contains $16, 96, 76, 56, 36$.

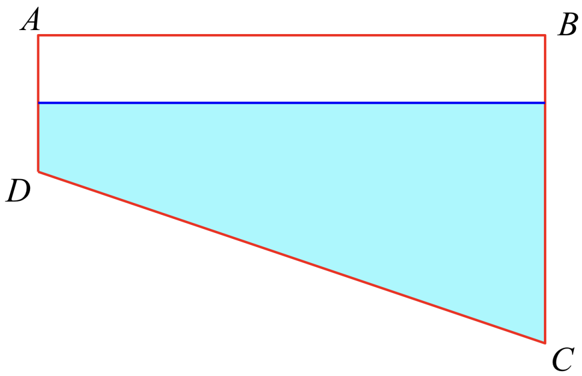
When $u = 5$, we find that S contains 25.

The sum of the elements in S is 975.

Question J-20 [5 points]

The horizontal cross section of a swimming pool is a rectangle, while the vertical cross section is a trapezium. The figure below shows the vertical cross section of the swimming pool. The shallow side of the pool AD has height 100 cm, while the deeper side BC has height 180 cm. The pool is originally full of water. After the water level drops by k cm, the pool only left with 77% of water. Find the value of $10k$.

一泳池的水平面是一长方形，直切面是梯形。下图所示为泳池的直切面。泳池浅的一边 AD 的高度是 100 cm，深的一边 BC 的高度是180 cm。开始时泳池的水是满的，水面下降了 k cm 后，泳池剩下 77% 的水。求 $10k$ 的值。



Answer: [322]

Solutions:

$$\frac{100 - k + 180 - k}{100 + 180} = \frac{77}{100}$$

$$10k = 322$$

Question J-21 [6 points]

If the first 10 terms of the sequence of odd positive integers $1, 3, 5, 7, \ldots$ are merged, we obtain the 15-digit number 135791113151719. If the first n terms of the sequence are merged, we obtain a 2017-digit number N. Find the last three digits of N.

将正奇数数列 $1, 3, 5, 7, \ldots$ 的前面 10 项连在一起，就得到 135791113151719 这个 15 位数。如果将前面 n 项连在一起会得到一个 2017 位数 N，求 N 的最后三位数。

Answer: [285]

Solutions:

Merging the 5 odd numbers from 1 to 9 give 5 digits.

Merging the 45 odd numbers from 11 to 99 gives 90 digits.

Merging the 450 odd numbers from 101 to 999 gives 1350 digits.

Merging the $n - 500$ odd numbers from 1001 to $2n - 1$ will give $4(n - 500)$ digits.

$$5 + 90 + 1350 + 4(n - 500) = 2017$$

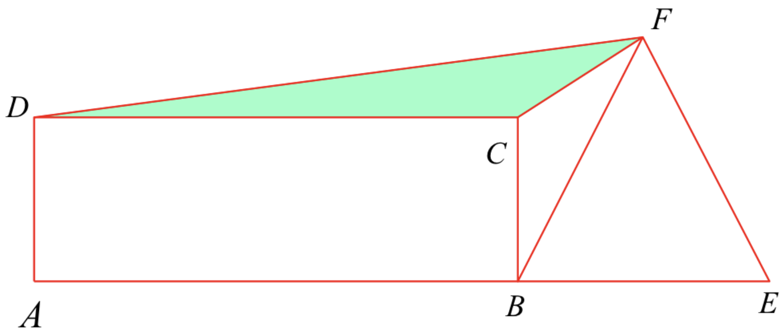
Therefore, $n = 643$.

The last four digits of N is $2 \times 643 - 1 = 1285$.

Question J-22 [6 points]

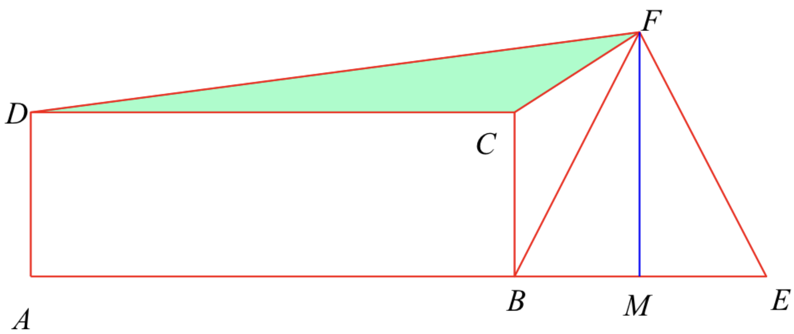
In the figure shown below, ABE is a straight line. $ABCD$ is a rectangle with $AB = 74$ and $AD = 34$. ΔBEF is an equilateral triangle. ΔBCF is an isosceles triangle. Find the area of ΔCDF.

下图中，ABE 是一直线。$ABCD$ 是长方形，$AB = 74$，$AD = 34$。ΔBEF 是等边三角形，ΔBCF 是一等腰三角形。求 ΔCDF 的面积。



Answer: [629]

Solutions:



Let $a = AB = 74$ and $b = AD = 34$.

$\angle CBF = 30^\circ$, $BF = \sqrt{3}b$, $FM = \frac{3}{2}b$.

Using CD as base, the height of ΔCDF is $FM - BC = \frac{b}{2}$.

Hence, the area of ΔCDF is

$$\frac{1}{2} \times a \times \frac{b}{2} = 629.$$

Question J-23 [6 points]

Given that a, b, c are three positive integers. The greatest common divisor of a and b is 540, the greatest common divisor of a and c is 504, and the least common multiple of b and c is 83160. If m is the greatest common divisor of b and c, find the largest possible value of m.

已知 a，b，c 三数是正整数，a 与 b 的最大公因数是 540，a 与 c 的最大公因数是 504，b 与 c 的最小公倍数是 83160。若 m 是 b 与 c 的最大公因数，求 m 的最大可能值。

Answer: [396]

Solutions:

b is a multiple of $540 = 2^2 \times 3^3 \times 5$.

c is a multiple of $504 = 2^3 \times 3^2 \times 7$.

a is a multiple of $2^3 \times 3^3 \times 5 \times 7$.

Therefore, b is not divisible by 2^3 and 7; c is not divisible by 3^3 and 5.

b and c are factors of $83160 = 2^3 \times 3^3 \times 5 \times 7 \times 11$. The maximum possible value of m is $2^2 \times 3^2 \times 11 = 396$, when

$$\begin{aligned} a =& 2^3 \times 3^3 \times 5 \times 7 \\ b =& 2^2 \times 3^3 \times 5 \times 11 \\ c =& 2^3 \times 3^2 \times 7 \times 11 \end{aligned}$$

Question J-24 [6 points]

Let $S = \{1, 3, 5, \ldots, 99\}$ be the set containing all positive odd integers less than 100. For a set A that is a subset of S, define $f(A)$ to be the sum of the elements in A. What is the maximum number of elements that A can have if $f(A) = 1900$?

设 $S = \{1, 3, 5, \ldots, 99\}$ 为小于 100 的正奇数集合。对于 S 的子集合 A，定义 $f(A)$ 为 A 中所有元素的和。若 $f(A) = 1900$, A 最多有几个元素?

Answer: [42]

Solutions:

Notice that if A has m elements, then

$$f(A) \geq 1 + 3 + \ldots + 2m - 1 = m^2.$$

For $f(A) = 1900$, $m \leq \sqrt{1900} = 43.59$, and A must have an even number of elements.

Hence, A cannot have more than 42 elements.

The set $A = \{1, 3, 5, \ldots, 65\} \cup \{75\} \cup \{85, 87, 89, 91, 93, 95, 97, 99\}$ has exactly 42 elements and the sum of the elements is 1900.

Question J-25 [6 points]

Given that the degrees of the four internal angles of a quadrilateral are a, $a+d$, $a+2d$, $a+3d$, where a and d are positive integers. Find the largest possible value of d.

已知一个四边形的四个内角的度数分别为 a，$a+d$，$a+2d$，$a+3d$，其中 a 与 d 都是正整数。求 d 的最大可能值。

Answer: [58]

Solutions:

$$a + a + d + a + 2d + a + 3d = 360$$

$$2a + 3d = 180$$

The positive integer solutions to $2a + 3d = 180$ is $a = 90 - 3k$, $d = 2k$, with $k \leq 29$. Hence, d is at most 58.

Question J-26 [8 points]

Given that a triangle is such that each of the three side lengths can only be one of the six integers

$$7 \quad 10 \quad 13 \quad 16 \quad 19 \quad 22$$

How many such triangles are there?

已知一三角形的三边长都必须是以下六个数之中的一个:

$$7 \quad 10 \quad 13 \quad 16 \quad 19 \quad 22$$

有多少个这样的三角形?

Answer: [49]

Solutions:

If the three sides lengths are the same, there are six possible cases.

If two sides are the same, but the third side is different, assume that the three side lengths are a, a and b with $b \neq a$.

If $a = 7$, b can be 10 or 13, two cases.

If $a = 10$, b can be 7, 13, 16 or 19, four cases.

If $a = 13$, b can be 7, 10, 16, 19 and 22, five cases.

If $a = 16$, b can be 7, 10, 13, 19, 22, five cases.

If $a = 19$, b can be 7, 10, 13, 16, 22, five cases.

If $a = 22$, b can be 7, 10, 13, 16, 19, five cases.

If all three sides have different lengths, we assume that the three side lengths are a, b and c with $a < b < c$.

If $(a, b) = (7, 10)$, c can only be 13, 16, two cases.

If $(a, b) = (7, 13)$, c can only be 16, 19, two cases.

If $(a, b) = (7, 16)$, c can only be 19, 22, two cases.

If $(a, b) = (7, 19)$, c can only be 22, one case.

If $(a, b) = (10, 13)$, c can only be 16, 19, 22, three cases.

If $(a, b) = (10, 16)$, c can only be 19, 22, two cases.

If $(a, b) = (10, 19)$, c can only be 22, one case.

If $(a, b) = (13, 16)$, c can only be 19, 22, two cases.

If $(a, b) = (13, 19)$, c can only be 22, one case.

If $(a, b) = (16, 19)$, c can only be 22, one case.

Hence, the total number of possible such triangles is 49.

Question J-27 [8 points]

A right-angled triangle is called perfect if the lengths of all three sides are integers, and one of the sides has length 100. How many perfect right-angled triangles are there?

如果一直角三角形三边的长都是整数，且其中一边的长等于100，我们称这三角形为完美三角形。有多少个直角三角形是完美的?

Answer: [9]

Solutions:

Let the lengths of the triangle be a, b and c, with $c^2 = a^2 + b^2$. We discuss two cases.

Case 1: $c \neq 100$.

Without loss of generality, we can assume that $a = 100$.

Then

$$(c-b)(c+b) = 100^2 = 2^4 5^4.$$

Notice that $c - b$ and $c + b$ must be both even or both odd. Since their product is even, they must be both even.

Let $c = 2c_1$ and $b = 2b_1$. Then

$$(c_1 - b_1)(c_1 + b_1) = 2^2 \times 5^4.$$

For any two integers m and n with $1 \leq m < n$ such that $mn = 2^2 \times 5^4$, we have a perfect triangle with $b = n - m$, $c = n + m$.

Since $2^2 \times 5^4$ has 3×5 positive factors, there are 7 perfect triangles in this case.

Case 2: $c = 100$.

Then $a^2 + b^2 = 100^2$.

Since a^2 and b^2 can only be congruent to 0 or 1 modulo 4, but 100^2 is congruent to 0 modulo 4, we must have a and b both even.

We can argue in the same way that a and b are divisible by 4.

Let $a = 4a_1$, $b = 4b_1$, then

$$a_1^2 + b_1^2 = 25^2,$$

and a_1 and b_1 must be one even and one odd, and both must be less than 25.

Without loss of generality, assume that a_1 is even and b_1 is odd.

Let $d = \gcd(a_1, b_1)$. Then $d \mid 25$. So d can only be 1, 5, cannot be 25.

Case 2A: $d = 1$. Then a_1 and b_1 are both not divisible 5 and

$$a_1^2 = (25 - b_1)(25 + b_1).$$

If $25 - b_1$ and $25 + b_1$ have common factor larger than 2, then b_1 and 25 have common factor larger than 1, which is also a common factor of a_1. This gives a contradiction.

Hence, $\dfrac{25 - b_1}{2}$ and $\dfrac{25 + b_1}{2}$ must be relatively prime.

This implies that $25 - b_1 = 2m^2$ and $25 + b_1 = 2n^2$ with m and n relatively prime positive integers, and

$$m^2 + n^2 = 25.$$

From this, we can only have $m = 3$ and $n = 4$. So $b_1 = 7$, $a_1 = 24$.

Case 2B: $d = 5$. Let $a_1 = 5a_2$ and $b_1 = 5b_2$. Then

$$a_2^2 + b_2^2 = 5^2.$$

We can only have $a_2 = 4$, $b_2 = 3$.

Altogether, we have exactly 9 perfect triangles.

Question J-28 [8 points]

Let S be the set of positive integers less than 100. Given that any n-element subset of S contains two elements that are not relatively prime, find the smallest possible value of n.

设 S 是所有小于 100 的正整数的集合。已知 S 的任一个含有 n 个元素的子集合一定会有两个不互质的元素，求 n 的最小可能值。

Answer: [27]

Solutions:

Let A be the subset of S that contains 1 and all the prime numbers in S.

A has 26 elements.

Let the elements in A be $1 = a_1 < a_2 < a_3 < \ldots < a_{26}$.

Define $A_1 = \{1\}$, and for $2 \leq k \leq 26$, let

$$A_k = \{m \in S \,|\, m \text{ is divisible by } a_k\}.$$

Then

$$S = \bigcup_{k=1}^{26} A_k.$$

Notice that any two elements of A are coprime.

If B is a subset of S that contains 27 elements, two of them must come from the same A_k for some k. These two elements of B are not coprime.

This shows that the smallest possible value of n is 27.

Question J-29 [8 points]

Let

$$S = \{n \mid n \text{ is a positive integer less than 100000, the product of digits of } n \text{ is } 180\}$$

Find the number of elements in S.

设

$$S = \{n \mid n \text{ 是小于 100000 的正整数且 } n \text{ 的各位数字的乘积是 } 180\}$$

求 S 中元素的个数。

Answer: [453]

Solutions:

$$180 = 2^2 \times 3^2 \times 5$$

If n is a three-digit number, there are two cases.

- The three digits can be 4, 9 and 5. There are 6 such integers.
- The three digits can be 6, 6, 5. There are 3 such integers.

If n is a four-digit number, there are five cases.

- The four digits can be 1, 4, 9 and 5. There are 24 such integers.
- The four digits can be 1, 6, 6 and 5. There are 12 such integers.
- The four digits can be 2, 2, 5, 9. There are 12 such integers.
- The four digits can be 4, 3, 3, 5. There are 12 such integers.
- The four digits can be 6, 2, 3, 5. There are 24 such integers.

If n is a five-digit number, there are six cases.

- The five digits can be 1, 1, 4, 9 and 5. There are 60 such integers.
- The five digits can be 1, 1, 6, 6 and 5. There are 30 such integers.
- The five digits can be 1, 2, 2, 5, 9. There are 60 such integers.
- The five digits can be 1, 4, 3, 3, 5. There are 60 such integers.

- The five digits can be 1, 6, 2, 3, 5. There are 120 such integers.
- The five digits can be 2, 2, 3, 3, 5. There are 30 such integers.

S has 453 elements.

Question J-30 [8 points]

Given that a, b, x and y are real numbers such that

$$\begin{aligned} ax - by &= 7 \\ a^2x^3 - b^2y^3 &= 22 \\ a^3x^5 - b^3y^5 &= 69 \\ a^4x^7 - b^4y^7 &= 216 \\ a^5x^9 - b^5y^9 &= K \end{aligned}$$

Find K.

已知 a，b，x 及 y 是实数且

$$\begin{aligned} ax - by &= 7 \\ a^2x^3 - b^2y^3 &= 22 \\ a^3x^5 - b^3y^5 &= 69 \\ a^4x^7 - b^4y^7 &= 216 \\ a^5x^9 - b^5y^9 &= K \end{aligned}$$

求 K。

Answer: [675]

Solutions:

$$\begin{aligned} (a^2x^3 - b^2y^3)(ax^2 + by^2) &= a^3x^5 - b^3y^5 + abx^2y^2(ax - by) \\ (a^3x^5 - b^3y^5)(ax^2 + by^2) &= a^4x^7 - b^4y^7 + abx^2y^2(a^2x^3 - b^2y^3) \\ (a^4x^7 - b^4y^7)(ax^2 + by^2) &= a^5x^9 - b^5y^9 + abx^2y^2(a^3x^5 - b^3y^5) \end{aligned}$$

Let $u = ax^2 + by^2$, $v = abx^2y^2$. Then

$$\begin{aligned} 22u - 7v &= 69 \\ 69u - 22v &= 216 \\ 216u - 69v &= K \end{aligned}$$

Solving the first two equations give $u = 6$ and $v = 9$. Hence, $K = 675$.