厦门大学马来西亚分校

陈景润杯中学数学比赛

2017 年第 1 届陈景润杯中学数学比赛

~ 中阶组 ~

日期：2017 年 4 月 1 日

Date: 1st April 2017

时间： 下午 2 时至 4 时

Time: 2:00 p.m. to 4:00 p.m.

考生须知

Instructions and Information

1. 本试卷共有 30 题。
   This paper contains 30 questions.
   - 第 1 至第 10 题，选择题， 每题 4 分。
     Question 1 to Question 10, multiple choice questions, each question carries 4 marks.
   - 第 11 至第 30 题，问答题，每题请填入一个-1000至 1000 之间的整数为答案。
     Question 11 to Question 30, short questions. For each question, please fill in an integer between -1000 and 1000 as answer.
     - 第 11 至第 20 题每题 5 分。
       Question 11 to Question 20, each question carries 5 marks.
     - 第 21 至第 25 题每题 6 分。
       Question 21 to Question 25, each question carries 6 marks.
     - 第 26 至第 30 题每题 8 分。
       Question 26 to Question 30, each question carries 8 marks.
2. 请在答案纸内适当的空格中用 2B 铅笔清楚的写出每题的答案。对于选择题，只需填写 A，B，C，D 或 E 作为答案。每题只能填入一个答案，否则以答错论。
   Please use 2B lead pencils to put your answer to each question in the appropriate space provided on the answer sheet. For a multiple choice question, you only need to put in A, B, C, D or E as answer. You can only give an answer to each question, otherwise it is considered that you answer that question incorrectly.
3. 所有的图形并没有按照比例作图，只作为辅助之用。
   All the diagrams are not drawn to scales. They are intended only as aids.
4. 不许使用计算器，数学工具，手机或其他计算器。
   No calculators, maths stencils, mobile phones or other calculating aids are permitted.
5. 在答案纸上清楚写上姓名，考生编号，学校名称及在学年级。
   Write your name, IC number, school name and year of study clearly on the answer sheet.
6. 在监考老师宣布比赛开始之后，才可以翻开此考卷开始作答。
   You can only open this question booklet to start answering questions after the invigilator announce the beginning of the competition.

~~ 说明 ~~

~~ Notes ~~

在这份试卷中，$\lfloor x \rfloor$表示小于或等于x的最大整数。

例如：$\lfloor 2 \rfloor = 2$，$\lfloor -2 \rfloor = -2$，$\lfloor 2.6 \rfloor = 2$，$\lfloor -2.6 \rfloor = -3$。

In this paper, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

For example, $\lfloor 2 \rfloor = 2$, $\lfloor -2 \rfloor = -2$, $\lfloor 2.6 \rfloor = 2$, $\lfloor -2.6 \rfloor = -3$.

第 1 至第 10 题，选择题，每题 4 分。

Question 1 to Question 10, multiple choice questions, each question carries 4 marks.

1. 求4567^{4567}的个位数。

   Find the last digit of 4567^{4567}.

   A. 1 B. 3 C. 5 D. 7 E. 9

2. 若$y = (1-x)(3-x)(1+x)(3+x)$，其中x是实数，求y的最小值。

   If $y = (1-x)(3-x)(1+x)(3+x)$, where x is a real number, find the minimum value of y.

   A. -16 B. -10 C. 9 D. 10 E. 16

3. 若 a与b是整数且$\sqrt{11-6\sqrt{2}}$是方程式$x^2+ax+b=0$的根，求$a+b$的值。

   If a and b are integers, and $\sqrt{11-6\sqrt{2}}$ is a root of the equation $x^2+ax+b=0$, find the value of $a+b$.

   A. -1 B. 1 C. 3 D. 7 E. 13

4. 若一长方形的长增加20%，宽减少10%，则它的面积增加多少？

   If the length of a rectangle is increased by 20%, and its width is decreased by 10%, by how much is its area increased?

   A. 8%
   B. 10%
   C. 30%
   D. 32%
   E. 无法确定 Inconclusive

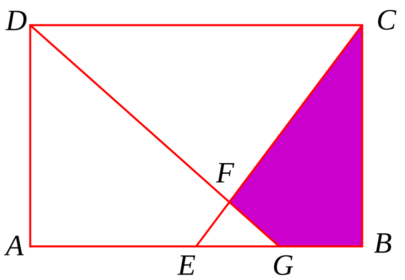
5. 已知一正n边形有 119 条对角线，求n的值。

If a regular n-gon has 119 diagonals, find the value of n.

A. 14　　B. 15　　C. 16　　D. 17　　E. 18

6. 下图中，$ABCI$是长方形，点E是线段AB的中点，点F在直线EC上使得$EF:FC=1:5$。DFG是直线。求阴影部分的面积与长方形$ABCD$的面积之比。

In the figure, $ABCD$ is a rectangle. E is the midpoint of the line segment AB. F is a point on the line EC such that $EF:FC=1:5$. DFG is a straight line. Find the ratio of the area of the shaded region to the area of the rectangle $ABCD$.



A. $9:40$　　B. $7:30$　　C. $1:5$　　D. $5:24$　　E. $3:16$

7. 若a，b，c是实数使得$\frac{b+c}{3a}=\frac{a+c}{3b}=\frac{a+b}{3c}=r$，求$r$的值。

If a, b, c are real numbers such that $\frac{b+c}{3a}=\frac{a+c}{3b}=\frac{a+b}{3c}=r$, find the value of r.

A. $\frac{2}{3}$　　B. $\frac{1}{3}$　　C. $\frac{2}{3}$ 或 or $-\frac{1}{3}$　　D. $\frac{2}{3}$ 或 or $\frac{1}{3}$　　E. $-\frac{1}{3}$ 或 or $\frac{1}{3}$

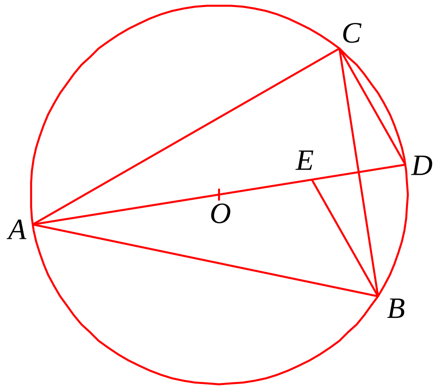
8. 设$S=\{1, 2, 3, 4, 5, 6\}$。A是S的子集，其所包含的奇数比偶数多。求满足这条件的子集合A的个数。

Let $S=\{1, 2, 3, 4, 5, 6\}$. A is a subset of S which contains more odd numbers than even numbers. Find the number of subsets A that satisfy this condition.

A. 21　　B. 22　　C. 24　　D. 30　　E. 32

9. 下图中，ΔABC 是等腰三角形，$AB = AC$，AD 是直径，点 O 是圆心，点 E 是线段 OD 的中点。若 $BE // CD$，$BC = 14$，求 OD^2。

In the figure below, ΔABC is an isosceles triangle, $AB = AC$, AD is a diameter, O is the center of the circle, E is the midpoint of the line segment OD. If $BE // CD$, $BC = 14$, find OD^2.



A. 112　　B. 105　　C. 98　　D. 91　　E. 84

10. 求 $\left\lfloor \frac{1^2}{3} \right\rfloor + \left\lfloor \frac{2^2}{3} \right\rfloor + \left\lfloor \frac{3^2}{3} \right\rfloor + \left\lfloor \frac{4^2}{3} \right\rfloor + \ldots + \left\lfloor \frac{30^2}{3} \right\rfloor$。

Find $\left\lfloor \frac{1^2}{3} \right\rfloor + \left\lfloor \frac{2^2}{3} \right\rfloor + \left\lfloor \frac{3^2}{3} \right\rfloor + \left\lfloor \frac{4^2}{3} \right\rfloor + \ldots + \left\lfloor \frac{30^2}{3} \right\rfloor$.

A. 3151　　B. 3149　　C. 3147　　D. 3145　　E. 3143

第 11 至第 20 题，问答题，每题 5 分。

Question 11 to Question 20, short questions, each question carries 5 marks.

11. 求 $\sqrt{499^2 + 999}$ 的值。

Find the value of $\sqrt{499^2 + 999}$.

12. 有多少个四位数能被 15 整除？
How many of the four-digit numbers are divisible by 15?

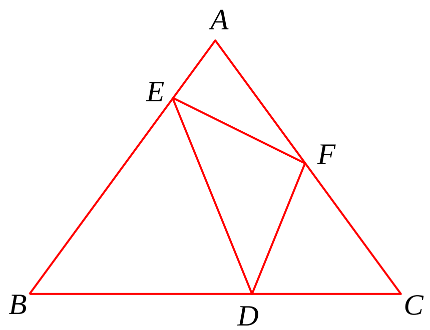
13. 已知 $\log_{10} 2 = 0.3010$，则 2^{360} 这个数有几位数?
Given that $\log_{10} 2 = 0.3010$, then how many digits does the number 2^{360} have?

14. 若 x， y 是实数，求函数 $f(x, y) = 9x^2 + 30xy + 25y^2 + 4x^2 + 4x + 18$ 的最小可能值。
If x, y are real numbers, find the minimum possible value of

$$f(x, y) = 9x^2 + 30xy + 25y^2 + 4x^2 + 4x + 18$$

15. 利民与丽芬两人在一环形跑道上以等速跑步。他们同时从同一地点出发并相背而跑，第一次相遇后，丽芬再跑了 9 分钟才回到起点。若利民跑一圈的时间是 10 分钟，丽芬跑一圈要多少时间？
Li-Min and Li-Fen were running on a circular track with constant speed. They departed from the same point at the same time, but went in opposite directions. After they met on the track for the first time, Li-Fen took another 9 minutes to return to the starting point. If Li-Min used 10 minutes to run one round, how long did it take for Li-Fen to run one round?

16. 下图中， ΔABC 是等腰三角形， $AB = AC$， $\angle A = 76°$。 $BD = BE$， $CD = CF$。若 $\angle DEF : \angle DFE = 7 : 9$， $\angle AEF = x°$， 求 x。
In the figure below, ΔABC is an isosceles triangle, $AB = AC$, $\angle A = 76°$, $BD = BE$, $CD = CF$. If $\angle DEF : \angle DFE = 7 : 9$, $\angle AEF = x°$, find x.

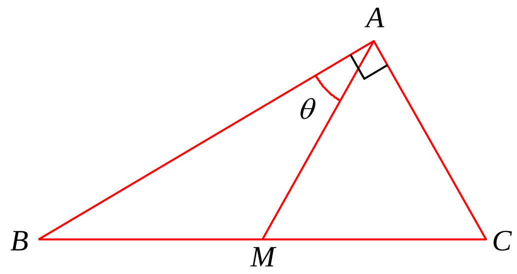


17. 设 n 是一个 正整数，它的各位数字之和是 2017。若 N 是所有满足这个条件的数字 n 中最小的一个，则 N 有几位数字？

Let n be a positive integer. The sum of the digits of n is 2017. If N is the smallest among all the integers n that satisfies the given condition, how many digits does N have?

18. 下图中，ΔABC 是直角三角形，$\angle A$ 是直角。M 是 BC 的中点，$\angle BAM=\theta$。若 $AM=15$，$AC=18$，求 $\dfrac{120}{\cos\theta}$。

In the figure below, ΔABC is a right-angled triangle, $\angle A$ is a right angle. M is the midpoint of BC, $\angle BAM=\theta$. If $AM=15$, $AC=18$, find $\dfrac{120}{\cos\theta}$.



19. 若 $S=\dfrac{1}{\frac{1}{2013}+\frac{2}{2014}+\frac{3}{2015}+\frac{4}{2016}+\frac{5}{2017}+\frac{4}{2018}+\frac{3}{2019}+\frac{2}{2020}+\frac{1}{2021}}$，求 $\lfloor S\rfloor$。

If $S=\dfrac{1}{\frac{1}{2013}+\frac{2}{2014}+\frac{3}{2015}+\frac{4}{2016}+\frac{5}{2017}+\frac{4}{2018}+\frac{3}{2019}+\frac{2}{2020}+\frac{1}{2021}}$, find $\lfloor S\rfloor$.

20. 求 $\left\lfloor\left(\sqrt{3}+1\right)^6\right\rfloor$。

Find $\left\lfloor\left(\sqrt{3}+1\right)^6\right\rfloor$.

第 21 至第 25 题，问答题，每题 6 分。

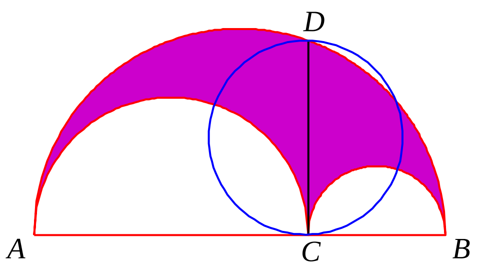
Question 21 to Question 25, short questions, each question carries 6 marks.

21. 若 x， y， z 是实数使得 $x^4y^2z^6=64$，求 $\frac{x^2}{2}+\frac{y^2}{4}+\frac{z^2}{3}$ 的最小可能值。

If x, y, z are real numbers such that $x^4y^2z^6=64$, find the minimum possible value of $\frac{x^2}{2}+\frac{y^2}{4}+\frac{z^2}{3}$.

22. 下图中，三个半圆两两互切。线段 CD 与直线 AB 互相垂直。设 S_1 是阴影部分的面积，S_2 是以线段 CD 为直径的圆的面积。求 $\left\lfloor \frac{120S_1}{S_2} \right\rfloor$。

In the figure below, the three semi-circles are mutually tangent to each other. The line segment CD is perpendicular to the line AB. Let S_1 be the area of the shaded region and S_2 the area of the circle with CD as diameter. Find $\left\lfloor \frac{120S_1}{S_2} \right\rfloor$.

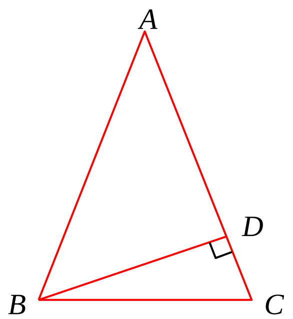


23. 已知 $\log_9 p=\log_{12} q=\log_{16}(6p+q)$。若 $\frac{q}{p}=x$，求 $\lfloor 10x \rfloor$。

Given that $\log_9 p=\log_{12} q=\log_{16}(6p+q)$. If $\frac{q}{p}=x$, find $\lfloor 10x \rfloor$.

24. 下图中，ΔABC 是等腰三角形，$AB = AC$，D 是点 B 到 AC 边上的垂足。若 AD 与 CD 的长度都是整数，$BD^2 = 85$，求 AC 长度的最小可能值。

In the following figure, ΔABC is an isosceles triangle, $AB = AC$. D is the foot of perpendicular from B to AC. If the lengths of AD and CD are integers, and $BD^2 = 85$, find the minimum possible length of AC.



25. 求 $1\underbrace{000\ldots000}_{2016\text{个}0}1$ 的最小的正质因数。

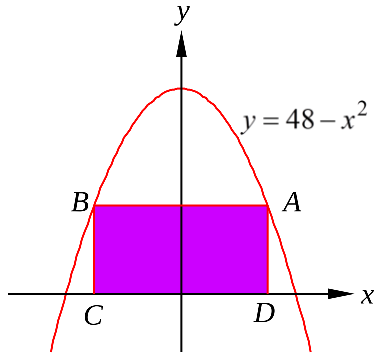
Find the smallest positive prime factor of $1\underbrace{000\ldots000}_{2016\ \text{0's}}1$.

第 26 至第 30 题，问答题，每题 8 分。

Question 26 to Question 30, short questions, each question carries 8 marks.

26. 如下图所示，长方形 $ABCD$ 的其中两个顶点在 x 轴上，另两个顶点在曲线 $y = 48 - x^2$ 上。求这个长方形的最大可能面积。

As shown in the figure below, two vertices of the rectangle $ABCD$ lie on the x-axis, and the other two vertices are on the curve $y = 48 - x^2$. Find the maximum possible area of this rectangle.



27. 若x， y是实数且$(x, y)\neq(0, 0)$，求

$$f(x, y)=\frac{12x^2+6xy+6y^2}{x^2+xy+y^2}$$

的最大可能值。

If x, y are real numbers such that $(x, y)\neq(0, 0)$, find the maximum possible value of

$$f(x, y)=\frac{12x^2+6xy+6y^2}{x^2+xy+y^2}$$

28. 设n是正整数。定义$A_n=\frac{30^n+15^n}{n!}$，其中$n!=1\times2\times3\times\ldots\times n$。求当$A_n$的值最大时$n$的值。

Let n be a positive integer and define $A_n=\frac{30^n+15^n}{n!}$, where $n!=1\times2\times3\times\ldots\times n$. Find the value of n when the value of A_n is largest.

29. 已知一三角形三边的长为a，b，c，其中a，b，c都是整数。若三角形的周长是P个单位，面积是A个平方单位，其中$P=2A$，求A的最大可能值。

Given that the three sides of a triangle have lengths a, b, c units respectively, where a, b, c are integers. If the perimeter is P units, the area is A square units, with $P=2A$, find the largest possible value of A.

30. 设n是小于1000的正整数且n^n+1可以被30整除，求n的最大可能值。

Let n be a positive integer that is less than 1000 such that n^n+1 is divisible by 30. Find the largest possible value of n.