Question M-01 [4 points]

In the decimal expansion of $\frac{3}{13}$, what is the 2020$^{\text{th}}$ digit after the decimal point?

将 $\frac{3}{13}$ 以小数形式表示，求在小数点后第 2020 位的数字。

Answer: [7]

Solutions:

$\frac{3}{13} = 0.230769230769.....$

2020 divided by 6 gives a remainder of 4. Hence, the 2020$^{\text{th}}$ digit is 7.

Question M-02 [4 points]

If a convex polygon has 1000 sides, what is the minimum number of obtuse angles that it has? [Each internal angle of a convex polygon is less than $180°$.]

如果一个凸多边形有 1000 个边，则它至少有几个内角是钝角？
[一个凸多边形的每个内角都必须小于 $180°$。]

Answer: [997]

Solutions:

The sum of the internal angle of the polygon is $998 \times 180°$. Assume that the internal angles are $A_1, A_2, \ldots, A_{1000}$, where $180° > A_1 \geq A_2 \geq \ldots A_k > 90° \geq A_{k+1} \geq \ldots \geq A_{1000}$. In other words, there are k obtuse angles. Then

$$998 \times 180° = A_1 + A_2 + \ldots + A_k + A_{k+1} + \ldots + A_{1000}$$

$$< k \times 180° + (1000 - k) \times 90°$$

$$1996 < 2k + 1000 - k$$

$$k > 996$$

Hence, there must be at least 997 obtuse angles. For example, we can have 997 internal angles that have degree $179.91°$, and the remaining 3 internal angles have degree $89.91°$.

Question M-03 [4 points]

If the first 10 terms of the sequence of odd positive integers $1, 3, 5, 7, \ldots$ are merged, we obtain the 15-digit number 135791113151719. If the first n terms of the sequence are merged, we obtain a 2017-digit number. Find the value of n.

将正奇数数列 $1, 3, 5, 7, \ldots$ 的前 10 项连在一起，就得到 135791113151719 这个 15 位数。如果将前 n 项连在一起会得到一个 2017 位数，求 n 的值。

Answer: [643]

Solutions:

Merging the 5 odd numbers from 1 to 9 give 5 digits.

Merging the 45 odd numbers from 11 to 99 gives 90 digits.

Merging the 450 odd numbers from 101 to 999 gives 1350 digits.

Merging the $n - 500$ odd numbers from 1001 to $2n - 1$ will give $4(n - 500)$ digits.

$$5 + 90 + 1350 + 4(n - 500) = 2017$$

Therefore, $n = 643$.

Question M-04 [4 points]

Find the last three digits of $\left(2020^{2020}-1\right)^{2020}$.

求 $\left(2020^{2020}-1\right)^{2020}$ 的最后三位数。

Answer: [001]

Solutions:

$$\begin{aligned} 2020^{2020} = 202^{2020} \times 10^{2020} &\equiv 0 \mod 1000 \\ \left(2020^{2020}-1\right)^{2020} \equiv (-1)^{2020} = 1 &\mod 1000 \end{aligned}$$

Question M-05 [4 points]

The hypotenuse of a right-angled triangle has length 53, and the lengths of the other two sides differ by 1. Find the area of the triangle.

一直角三角形的斜边长 53，另外两个边的长相差 1。求此三角形的面积。

Answer: [702]

Solutions:

Let the two non-hypotenuse sides of the triangle be a and $a+1$. Then

$$a^2 + a^2 + 2a + 1 = 53^2$$

$$2a(a+1) = 53^2 - 1 = 52 \times 54$$

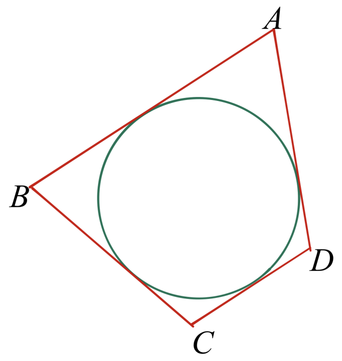
The area of the triangle is

$$\frac{a(a+1)}{2} = \frac{52 \times 54}{4} = 702.$$

Question M-06 [4 points]

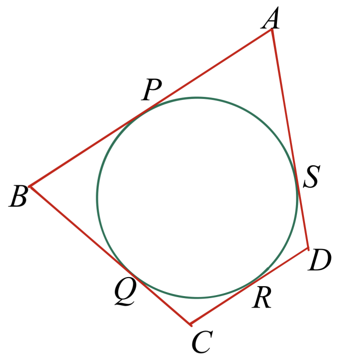
In the figure shown below, the lines AB, BC, CD and DA are tangents to the circle. If $AB = 101$, $BC = 97$, $CD = 67$, find AD.

下图中，直线 AB，BC，CD 及 DA 都与圆相切。若 $AB = 101$，$BC = 97$，$CD = 67$，求 AD。



Answer: [71]

Solutions:



Let P, Q, R and S be respectively the points of contact.

Let $AP = AS = a$, $BP = BQ = b$, $CQ = CR = c$ and $DR = DS = d$.

$$a + b = 101$$
$$b + c = 97$$
$$c + d = 67$$

Hence,

$$AD = d + a = 101 + 67 - 97 = 71$$

Question M-07 [4 points]

Given that n is a positive integer. If there are integers a and b such that $n = 2109a + 5928b$, find the smallest possible value of n.

已知 n 是一正整数且存在整数 a 和 b 使得 $n = 2109a + 5928b$。求 n 的最小可能值。

Answer: [57]

Solutions:

Using Euclidean algorithm,

$$\begin{aligned} 5928 &= 2 \times 2109 + 1710 \\ 2109 &= 1 \times 1710 + 399 \\ 1710 &= 4 \times 399 + 114 \\ 399 &= 3 \times 114 + 57 \\ 114 &= 2 \times 57 \end{aligned}$$

57 is the greatest common divisor of 2109 and 5928.

By working backward, we find that there are integers a and b such that

$$2109a + 5928b = 57.$$

Conversely, since 2109 and 5928 are both divisible by 57, any n that can be written in the form $n = 2109a + 5928b$ must be a multiple of 57.

The smallest such positive integer n is 57.

Question M-08 [4 points]

Given that when the polynomial $f(x)$ is divided by $2x^2-191x-1111$, the remainder is $2x+50$. Find the remainder when $f(x)$ is divided by $x-101$.

已知多项式 $f(x)$ 除以 $2x^2-191x-1111$ 得余项 $2x+50$，求 $f(x)$ 除以 $x-101$ 的余数。

Answer: [252]

Solutions:

$$2x^2-191x-1111=(2x+11)(x-101)$$
$$f(x)=q(x)(2x+11)(x-101)+2x+50$$

When $f(x)$ is divided by $x-101$, the remainder is $f(101)=252$.

Question M-09 [4 points]

Given a is a positive integer such that the minimum value of $f(x) = 3x^2 + ax + 10000$ is 253, find the value of a.

已知 a 是一正整数且 $f(x) = 3x^2 + ax + 10000$ 的最小值是 253，求 a 的值。

Answer: [342]

Solutions:

When $x = -\frac{a}{6}$, $f(x)$ has minimum value. Hence,

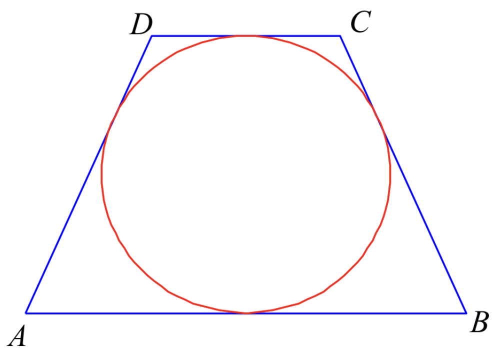
$$\frac{a^2}{12} - \frac{a^2}{6} + 10000 = 253$$
$$a^2 = 12 \times 9747 = 2^2 \times 3^4 \times 19^2$$
$$a = 342$$

There is a mistake in the original version of the question. Therefore students answering 342, or 342 and -342, are all given full marks.

Question M-10 [4 points]

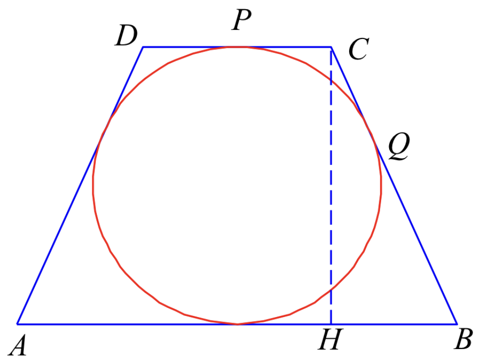
As shown in the figure below, a circle is inscribed in the trapezium $ABCD$. Given that $AD = BC = 65$, and the area of the circle is 784π. Find the length of AB.

如下图所示，一圆内切于梯形 $ABCD$ 中。已知 $AD = BC = 65$ 且圆的面积为 784π，求 AB 的长。



Answer: [98]

Solutions:



The radius of the circle is 28. Let $CP = a$.

$$
\begin{aligned}
CH &= 56, \\
BH &= \sqrt{65^2 - 56^2} = 33, \\
CQ &= CP = a, \\
BQ &= \frac{1}{2}AB = a + 33.
\end{aligned}
$$

Then

$$a + a + 33 = 65$$

and hence $a = 16$.

The length of AB is 98.

Question M-11 [5 points]

A line separates the plane into 2 regions. Now the plane is separated into N regions by 67 lines, no two of which are parallel. Find the smallest possible value of N.

一条直线将平面分成两个区域。现平面上有 67 条直线，没有两条直线互相平行。若这些直线将平面分成 N 个区域，求 N 的最小可能值。

Answer: [134]

Solutions:

Since no two lines are parallel, any two lines will intersect.

To have minimum number of regions created, all the lines should intersect at a single point.

In this case, there are $2 \times 67 = 134$ regions.

Question M-12 [5 points]

How many ways are there to divide twelve students into two groups of six each?

有多少种方法可以将 12 位学生分成两组，每组 6 人?

Answer: [462]

Solutions:

Since the two groups are indistinguishable, the number of ways is

$$\frac{1}{2}\binom{12}{6} = 462.$$

Due to the slight ambiguity in the question, any students answering 924 would also be given full marks.

Question M-13 [5 points]

Given that $a_1, a_2, a_3, \ldots$ is a sequence such that $a_n - a_{n-1} = n$ for all $n \geq 2$. If the 40th term of the sequence is 957, find the 20th term of the sequence.

已知 $a_1, a_2, a_3, \ldots$ 是一数列，当 $n \geq 2$ 时，$a_n - a_{n-1} = n$。若此数列的第 40 项是 957，求此数列的第 20 项。

Answer: [347]

Solutions:

$$a_2 - a_1 = 2$$
$$a_3 - a_2 = 3$$
$$\vdots$$
$$a_n - a_{n-1} = n$$
$$a_n - a_1 = 2 + 3 + \ldots + n = \frac{n(n+1)}{2} - 1$$
$$a_{40} = a_1 - 1 + 820 = 957$$
$$a_1 = 138$$
$$a_{20} = a_1 - 1 + 210 = 347$$

Question M-14 [5 points]

The 2020 students in a school are divided into 7 groups to compete in a game. No two groups have the same number of students. What is the minimum number of students in the largest group?

一间学校有 2020 位学生。如果将这些学生分成 7 组来进行运动会，没有两组有相同的人数，那么人数最多的组至少有几个人？

Answer: [292]

Solutions:

Let the number of students in each group be $m_1, m_2, m_3, m_4, m_5, m_6, m_7$ with

$$m_1 > m_2 > \ldots > m_7$$

This implies that $m_2 \leq m_1 - 1$, $m_3 \leq m_2 - 1 \leq m_1 - 2$, …, $m_7 \leq m_1 - 6$.

Since $m_1 + m_2 + m_3 + m_4 + m_5 + m_6 + m_7 = 2020$, we have

$$7m_1 - 21 \geq 2020.$$

Hence, m_1 is at least 292. In fact, we can have the 7 groups have 292, 291, 290, 289, 288, 287, 283 students respectively.

Question M-15 [5 points]

If k is a positive integer and both the roots of the equation $x^2 - kx + 2020 = 0$ are integers, find the smallest possible value of k.

若 k 是正整数且方程式 $x^2 - kx + 2020 = 0$ 的两个根都是整数，求 k 的最小可能值。

Answer: [121]

Solutions:

The two roots of $x^2 - kx + 2020 = 0$, call them a and b must be positive.

We have $a + b = k$, $ab = 2020 = 2^2 \times 5 \times 101$.

Without loss of generality, assume that $a < b$. Then (a, b) can be one of the following 6 cases:

$$(1, 2020), \quad (2, 1010), \quad (4, 505), \quad (5, 404), \quad (10, 202), \quad (20, 101).$$

Hence, the smallest possible value of k is 121.

Question M-16 [5 points]

Given that a and b are positive numbers such that $2\sqrt{a}\left(\sqrt{a}+50\sqrt{b}\right)=101\sqrt{b}\left(\sqrt{a}+201\sqrt{b}\right)$. Find

$$\frac{14a-9\sqrt{ab}+b}{a-10000b}.$$

已知 a 与 b 是正数且 $2\sqrt{a}\left(\sqrt{a}+50\sqrt{b}\right)=101\sqrt{b}\left(\sqrt{a}+201\sqrt{b}\right)$。求

$$\frac{14a-9\sqrt{ab}+b}{a-10000b}.$$

Answer: [706]

Solutions:

$$2\sqrt{a}\left(\sqrt{a}+50\sqrt{b}\right)=101\sqrt{b}\left(\sqrt{a}+201\sqrt{b}\right)$$
$$2a-\sqrt{ab}-20301b=0$$
$$(2\sqrt{a}+201\sqrt{b})(\sqrt{a}-101\sqrt{b})=0$$

Since $\sqrt{a}$ and $\sqrt{b}$ must be positive, we find that $\sqrt{a}=101\sqrt{b}$.

Hence,

$$\frac{14a-9\sqrt{ab}+b}{a-10000b}=\frac{101^2\times 14b-9\times 101b+b}{101^2b-10000b}=706$$

Question M-17 [5 points]

A school has four clubs whose members are students in this school. Each club has 177 members. Every two clubs have 52 common members. Every three clubs have 23 common members. There are exactly 11 students that join all four clubs. How many students join exactly one club?

已知一间学校有 4 个学会，每个学会的会员都是该校的学生。每个学会有 177 位会员，每两个学会有 52 位共同会员，每三个学会有 23 位共同会员，恰有 11 位学生是这四个学会的共同会员。只参加一个学会的学生有几人?

Answer: [316]

Solutions:

Let A_1, A_2, A_3, A_4 be respectively the set of students in the club 1, club 2, club 3 and club 4.

$$\begin{aligned} n(A_1 \cap (A_2 \cup A_3 \cup A_4)) =& n(A_1 \cap A_2) + n(A_1 \cap A_3) + n(A_1 \cap A_4) \\ & - n(A_1 \cap A_2 \cap A_3) - n(A_1 \cap A_2 \cap A_4) - n(A_1 \cap A_3 \cap A_4) \\ & + n(A_1 \cap A_2 \cap A_3 \cap A_4) \\ =& 3 \times 52 - 3 \times 23 + 11 = 98 \end{aligned}$$

Hence, the number of students that join only club 1 is $177 - 98 = 79$.

By symmetry, the number of students that only join club 2 is also 79, same for club 3 and club 4.

Hence, the number of students that join exactly one club is $4 \times 79 = 316$.

Question M-18 [5 points]

Let P be the product of all the solutions of the equation

$$\sqrt{x^2+2x+700}-\sqrt{x^2+2x-99}=17.$$

Find the absolute value of P.

设 P 是方程式

$$\sqrt{x^2+2x+700}-\sqrt{x^2+2x-99}=17$$

所有的解的乘积，求 P 的绝对值。

Answer: [324]

Solutions:

Notice that $47 \times 17 = 799$, and

$$\left(\sqrt{x^2+2x+700}-\sqrt{x^2+2x-99}\right)\left(\sqrt{x^2+2x+700}+\sqrt{x^2+2x-99}\right)=799.$$

Hence,

$$\sqrt{x^2+2x+700}+\sqrt{x^2+2x-99}=47.$$

This implies that

$$\sqrt{x^2+2x+700}=32$$

and

$$x^2+2x-324=0.$$

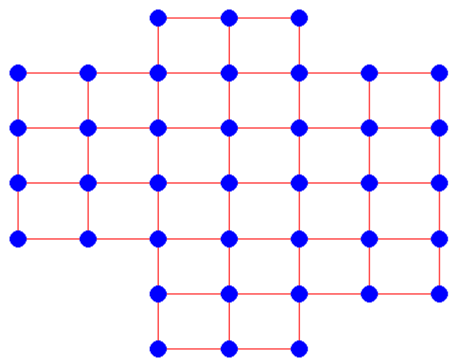
Conversely, if $x^2+2x=324$, x satisfies the given equation.

This shows that the equation has two solutions with product $P=-324$.

Question M-19 [5 points]

The figure below shows a network of 39 computers connected by 64 cables. If computer A is connected to computer B by a cable, and computer B is connected to computer C by a cable, then computer A and computer C are connected, even without a cable between them. At most how many of the cables can be removed for the computers to stay connected?

下图显示 39 台电脑被 64 条线路连接起来。如果有线路连接电脑 A 和电脑 B，也有线路连接电脑 B 和电脑 C，则电脑 A 和电脑 C 是互相连通的，即便他们之间没有线路连接。如果要这些电脑之间都相互连通，最多有几条线路可以被拔除？



Answer: [26]

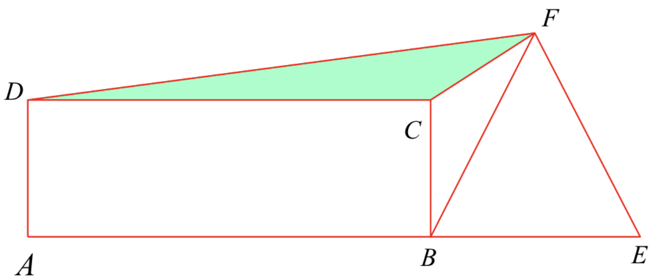
Solutions:

To have the 39 computers stay connected, we need at least 38 cables. Hence, at most 26 cables can be removed.

Question M-20 [5 points]

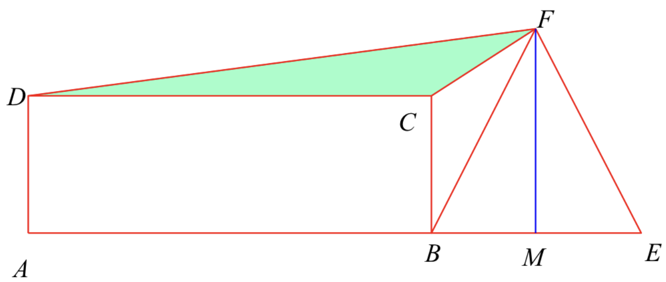
In the figure shown below, ABE is a straight line. $ABCD$ is a rectangle with $AB = 86$ and $AD = 38$. ΔBEF is an equilateral triangle. ΔBCF is an isosceles triangle. Find the area of ΔCDF.

下图中，ABE 是一直线。$ABCD$ 是长方形，$AB = 86$，$AD = 38$。ΔBEF 是等边三角形，ΔBCF 是一等腰三角形。求 ΔCDF 的面积。



Answer: [817]

Solutions:



Let $a = AB = 86$ and $b = AD = 38$.

$\angle CBF = 30°$, $BF = \sqrt{3}b$, $FM = \frac{3}{2}b$.

Using CD as base, the height of ΔCDF is $FM - BC = \frac{b}{2}$.

Hence, the area of ΔCDF is

$$\frac{1}{2} \times a \times \frac{b}{2} = 817.$$

Question M-21 [6 points]

Given that a is a positive constant. When x is a positive number, the minimum value of

$$x^6 + \frac{a^4}{x^6} + 14x^3 + 14\frac{a^2}{x^3}$$

is 702. Find the value of a^2.

已知 a 是正的常数。当 x 是正数时，

$$x^6 + \frac{a^4}{x^6} + 14x^3 + 14\frac{a^2}{x^3}$$

的最小可能值是 702，求 a^2 的值。

Answer: [169]

Solutions:

$$\begin{aligned} f(x) =& x^6 + \frac{a^4}{x^6} + 14x^3 + 14\frac{a^2}{x^3} \\ =& \left(x^3 + \frac{a^2}{x^3}\right)^2 - 2a^2 + 14\left(x^3 + \frac{a^2}{x^3}\right) \\ =& \left(x^3 + \frac{a^2}{x^3} + 7\right)^2 - 2a^2 - 49 \\ =& \left[\left(x\sqrt{x} - \frac{a}{x\sqrt{x}}\right)^2 + 2a + 7\right]^2 - 2a^2 - 49 \end{aligned}$$

When $x = \sqrt[3]{a}$, this expression has minimum value

$$(2a + 7)^2 - 2a^2 - 49 = 702.$$

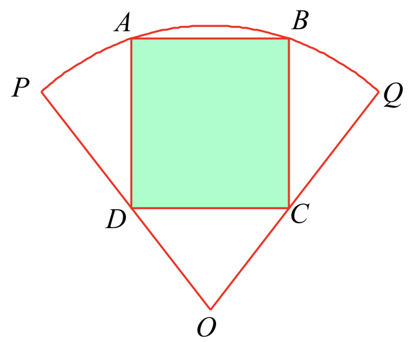
This gives $(a - 13)(a + 27) = a^2 + 14a - 351 = 0$.

Hence, $a = 13$ and $a^2 = 169$.

Question M-22 [6 points]

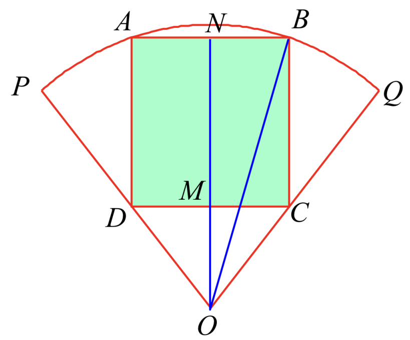
In the figure shown below, OPQ is a sector with $\angle POQ = 60°$. $ABCD$ is a square. If the area of the sector OPQ is 144π, and the area of the square $ABCD$ is S, find the largest integer less than S.

下图中，OPQ 是一扇形，$\angle POQ = 60°$。$ABCD$ 是正方形。已知扇形 OPQ 的面积是 144π，正方形 $ABCD$ 的面积为 S，求小于 S 的最大整数。



Answer: [231]

Solutions:



Let $OB = r$, $AB = a$. Then $CM = \frac{a}{2}$, $OM = \frac{\sqrt{3}}{2}a$.

$$\frac{1}{6}\pi r^2 = 144\pi$$
$$r^2 = 864$$

$$\left(\frac{a}{2}\right)^2 + \left(a + \frac{\sqrt{3}a}{2}\right)^2 = r^2$$

$$\left(\frac{1}{4} + 1 + \sqrt{3} + \frac{3}{4}\right) a^2 = r^2$$

$$S = a^2 = \frac{r^2}{2 + \sqrt{3}} = 864(2 - \sqrt{3}) = 231.5$$

Question M-23 [6 points]

If there are k distinct positive integers with sum smaller than 202020, find the largest possible value of k.

如果有 k 个不同的正整数，其和小于 202020，求 k 的最大可能值。

Answer: [635]

Solutions:

Let the k positive integers be $n_1, n_2, \ldots, n_k$. Then $n_k \geq k$.

$$202020 > n_1 + n_2 + \ldots + n_k \geq 1 + 2 + \ldots + k = \frac{k(k+1)}{2}$$

$$k^2 + k - 404040 < 0$$

$$k < \frac{-1 + \sqrt{1 + 4 \times 404040}}{2} = 635.14$$

The largest possible value of k is 635.

Question M-24 [6 points]

How many pairs of integers (x, y) satisfy the inequality $23|x| + 7|y| \leq 217$?

有多少对整数 (x, y) 满足不等式 $23|x| + 7|y| \leq 217$?

Answer: [589]

Solutions:

It can be divided into 7 cases:

Case 1 $x = y = 0$.

Case 2 $y = 0, x \neq 0$.

$|x| \leq \frac{217}{23} = 9.4$. There are 18 such points.

Case 3 $x = 0, y \neq 0$.

Then $|y| \leq \frac{217}{7}$. There are 62 such points.

Case 4 $x > 0, y > 0$.

Then $23x + 7y \leq 217$.

x can varies between 1 to 9. For fixed x, the number of positive integers y such that $23x + 7y \leq 217$ is $\left\lfloor \frac{217 - 23x}{7} \right\rfloor$.

This gives a total

$$\sum_{x=1}^{9} \left\lfloor \frac{217 - 23x}{7} \right\rfloor = 127$$

pairs of (x, y) for this case.

Case 5 $x < 0, y > 0$.

This is in one-to-one correspondence with Case 4. Hence, this case also gives 127 pairs.

Case 6 $x > 0, y < 0$.

This is in one-to-one correspondence with Case 4. Hence, this case also gives 127 pairs.

Case 7 $x < 0, y < 0$.

This is in one-to-one correspondence with Case 4. Hence, this case also gives 127 pairs.

Hence, the total number of pairs of integers (x, y) satisfying $23|x| + 7|y| \leq 217$ is 589.

Question M-25 [6 points]

Let $S = \{20x + 101y \,|\, x, y \text{ are nonnegative integers}\}$. Among the positive integers less than 10000, how many of them are not in the set S?

设 $S = \{20x + 101y \,|\, x, y \text{是非负的整数}\}$。在小于 10000 的正整数中，有多少个不在 S 中?

Answer: [950]

Solutions: Notice that if k is an integer, the integer solutions (x, y) to $20x + 101y = k$ can be written as

$$x = 101n - 5k, \quad y = k - 20n$$

for some integer n.

In order that x and y are nonnegative integers, we must have

$$20n \le k \le \frac{101}{5}n$$

for some nonnegative integers n.

Equivalently, there must be a nonnegative integer n so that

$$\frac{5}{101}k \le n \le \frac{k}{20}.$$

Notice that if $k > 2020$,

$$\frac{k}{20} - \frac{5}{101}k = \frac{k}{2020} > 1,$$

so there is always one integer n so that

$$\frac{5}{101}k \le n \le \frac{k}{20}.$$

Hence, we only need to count how many positive integers not larger than 2020 that are in S. This is the same as counting for each $1 \le n \le 100$, how many integers k are such that

$$20n \le k \le \frac{101}{5}n$$

Notice that when $n \le 100$,

$$\frac{101}{5}(n-1) < 20n,$$

and so no k would be double counted.

Hence, the number of positive integers k not larger than 2020 that are in S is

$$\sum_{n=1}^{100}\left(\left\lfloor \frac{101}{5}n \right\rfloor - 20n + 1\right) = \sum_{n=1}^{100}\left(\left\lfloor \frac{n}{5} \right\rfloor + 1\right) = 100 + 20 + \sum_{k=1}^{19} 5k = 1070.$$

Therefore, the number of positive integers not in S is $2020 - 1070 = 950$.

Question M-26 [8 points]

Let S be the set of positive integers less than 236. Given that any n-element subset of S contains two elements that are not relatively prime, find the smallest possible value of n.

设 S 是所有小于 236 的正整数的集合。已知 S 的任一个含有 n 个元素的子集合一定会有两个不互质的元素。求 n 的最小可能值。

Answer: [53]

Solutions:

Let A be the subset of S that contains 1 and all the prime numbers in S.

A has 52 elements.

Let the elements in A be $1 = a_1 < a_2 < a_3 < \ldots < a_{52}$.

Define $A_1 = \{1\}$, and for $2 \leq k \leq 52$, let

$$A_k = \{m \in S \,|\, m \text{ is divisible by } a_k\}.$$

Then

$$S = \bigcup_{k=1}^{52} A_k.$$

Notice that any two elements of A are coprime.

If B is a subset of S that contains 53 elements, two of them must come from the same A_k for some k. These two elements of B are not coprime.

This shows that the smallest possible value of n is 53.

Question M-27 [8 points]

Let

$$S = \{n \mid n \text{ is a positive integer less than } 100000\text{, the product of digits of } n \text{ is } 120\}$$

Find the number of elements in S.

设

$$S = \{n \mid n \text{ 是小于 } 100000 \text{ 的正整数且 } n \text{ 的各位数字的乘积是 } 120\}$$

求 S 中元素的个数。

Answer: [416]

Solutions:

$$120 = 2^3 \times 3 \times 5$$

If n is a three-digit number, there are two cases.

- The three digits can be 8, 3 and 5. There are 6 such integers.
- The three digits can be 4, 6, 5. There are 6 such integers.

If n is a four-digit number, there are four cases.

- The four digits can be 1, 8, 3 and 5. There are 24 such integers.
- The four digits can be 1, 4, 6 and 5. There are 24 such integers.
- The four digits can be 2, 4, 3, 5. There are 24 such integers.
- The four digits can be 2, 2, 6, 5. There are 12 such integers.

If n is a five-digit number, there are five cases.

- The five digits can be 1, 1, 8, 3 and 5. There are 60 such integers.
- The five digits can be 1, 1, 4, 6 and 5. There are 60 such integers.
- The five digits can be 1, 2, 4, 3, 5. There are 120 such integers.
- The five digits can be 1, 2, 2, 6, 5. There are 60 such integers.
- The five digits can be 2, 2, 2, 3, 5. There are 20 such integers.

S has 416 elements.

Question M-28 [8 points]

Let $S = \{1, 3, 5, \ldots, 35\}$ be the set containing all positive odd integers less than 36. For a set A that is a subset of S, define $f(A)$ to be the sum of the elements in A. If $f(A) = 36$, we say that A is perfect. Among all the subsets of S, how many of them are perfect?

设 $S = \{1, 3, 5, \ldots, 35\}$ 为小于 36 的正奇数集合。对于 S 的子集合 A，定义 $f(A)$ 为 A 中所有元素的和。若 $f(A) = 36$，A 就被称为完美的集合。S 的子集中，有几个是完美的?

Answer: [33]

Solutions:

Let A be a perfect subset of S. Notice that if A has m elements, then

$$f(A) \geq 1 + 3 + \ldots + (2m - 1) = m^2.$$

For $f(A) = 36$, $m \leq \sqrt{36} = 6$, and A must have an even number of elements.

Hence, A can only have 2, 4 or 6 elements.

If A has 6 elements, then A must be the set $\{1, 3, 5, 7, 9, 11\}$, only one possibility.

If A has 2 elements, then $A = \{2k_1 - 1, 2k_2 - 1\}$ with $1 \leq k_1 < k_2 \leq 18$ and $k_1 + k_2 = 19$. There are 9 possibilities.

The most difficult case is when A has 4 elements. In this case,

$$A = \{2k_1 - 1, 2k_2 - 1, 2k_3 - 1, 2k_4 - 1\}$$

with $1 \leq k_1 < k_2 < k_3 < k_4 \leq 18$ and $k_1 + k_2 + k_3 + k_4 = 20$. We can let

$$\begin{aligned} k_1 &= a + 1, \\ k_2 &= a + 1 + b + 1, \\ k_3 &= a + 1 + b + 1 + c + 1, \\ k_4 &= a + 1 + b + 1 + c + 1 + d + 1 \end{aligned}$$

The solutions for (k_1, k_2, k_3, k_4) are in one-to-one correspondence with the nonnegative integer solutions to

$$4a + 3b + 2c + d = 10.$$

We see that a can only be 0, 1 or 2.

When $a = 2$, $3b + 2c + d = 2$. Then we must have $b = 0$ and $2c + d = 2$. So we can only have

$(c, d) = (1, 0)$ or $(0, 2)$, 2 possibilities.

When $a = 1$, $3b + 2c + d = 6$. Then b can be 0, 1 or 2.

If $b = 2$, then we must have $(c, d) = (0, 0)$, only 1 possibility.

If $b = 1$, then $2c + d = 3$. Then we can only have $(c, d) = (0, 3)$ or $(1, 1)$, 2 possibilities.

If $b = 0$, then $2c + d = 6$. There are 4 possibilities for (c, d).

When $a = 0$, $3b + 2c + d = 10$. Then b can be 0, 1, 2 or 3.

If $b = 3$, then $2c + d = 1$. There is only 1 possibility for (c, d).

If $b = 2$, then $2c + d = 4$. There are 3 possibilities for (c, d).

If $b = 1$, then $2c + d = 7$. There are 4 possibilities for (c, d).

If $b = 0$, then $2c + d = 10$. There are 6 possibilities for (c, d).

Altogether, there are 33 prefect sets.

Question M-29 [8 points]

Let $P = 1 \times 3 \times \ldots \times 999$ be the product of the odd positive integers less than 1000. Find the largest integer k such that 3^k divides P.

设 $P = 1 \times 3 \times \ldots \times 999$ 是小于 1000 的正奇数的乘积。求最大的整数 k 使得 3^k 可以整除 P。

Answer: [251]

Solutions:

The multiples of 3 among all odd integers less than 1000 are

$$3, 9, 15, \ldots, 999$$

There are 167 of them.

The multiples of 9 among all odd integers less than 1000 are

$$9, 27, 45, \ldots, 999$$

There are 56 of them.

The multiples of 27 among all odd integers less than 1000 are

$$27, 81, \ldots, 999$$

There are 19 of them.

The multiples of 81 among all odd integers less than 1000 are

$$81, 243, 405, \ldots, 891$$

There are 6 of them.

The multiples of 243 among all odd integers less than 1000 are

$$243, 729$$

There are 2 of them.

The multiples of 729 among all odd integers less than 1000 are

$$729$$

There is 1 of them.

The largest k such that 3^k divides P is

$$167 + 56 + 19 + 6 + 2 + 1 = 251$$

Hence, the largest k such that 3^k divides P is 251.

Question M-30 [8 points]

Given that a, b, x and y are real numbers such that

$$\begin{aligned} ax - by =& 16 \\ a^2x^3 - b^2y^3 =& 23 \\ a^3x^5 - b^3y^5 =& 33 \\ a^4x^7 - b^4y^7 =& 48 \\ a^5x^9 - b^5y^9 =& K \end{aligned}$$

Find K.

已知 a，b，x 及 y 是实数且

$$\begin{aligned} ax - by =& 16 \\ a^2x^3 - b^2y^3 =& 23 \\ a^3x^5 - b^3y^5 =& 33 \\ a^4x^7 - b^4y^7 =& 48 \\ a^5x^9 - b^5y^9 =& K \end{aligned}$$

求 K。

Answer: [63]

Solutions:

$$\begin{aligned} (a^2x^3 - b^2y^3)(ax^2 + by^2) =& a^3x^5 - b^3y^5 + abx^2y^2(ax - by) \\ (a^3x^5 - b^3y^5)(ax^2 + by^2) =& a^4x^7 - b^4y^7 + abx^2y^2(a^2x^3 - b^2y^3) \\ (a^4x^7 - b^4y^7)(ax^2 + by^2) =& a^5x^9 - b^5y^9 + abx^2y^2(a^3x^5 - b^3y^5) \end{aligned}$$

Let $u = ax^2 + by^2$, $v = abx^2y^2$. Then

$$\begin{aligned} 23u - 16v =& 33 \\ 33u - 23v =& 48 \\ 48u - 33v =& K \end{aligned}$$

Solving the first two equations give $u = -9$ and $v = -15$. Hence, $K = 63$.