厦门大学马来西亚分校

陈景润杯中学数学比赛

CHEN JINGRUN'S CUP SECONDARY SCHOOL MATHEMATICS COMPETITION 2020

INTERMEDIATE CATEGORY

**** 中阶组 ****

Date: 22nd August 2020　　Time: 2:00 pm to 4:30 pm

日期：2020 年 8 月 22 日　　时间：下午 2 时至下午 4 时半

Instructions and Information

考生须知

1. This paper contains 30 questions. Each question is a short question which carries 4 to 8 points.
   本试卷有 30 道题目，每题都是问答题，分数介于 4 分至 8 分之间。

2. The answer for each question is an integer.
   每题的答案是一个整数。

3. All the diagrams are not drawn to scale. They are intended as aids only.
   所有的图形并没有按照比例作图，只作为辅助之用。

4. In this paper, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.
   For example, $\lfloor 2 \rfloor = 2$, $\lfloor -2 \rfloor = -2$, $\lfloor 2.6 \rfloor = 2$, $\lfloor -2.6 \rfloor = -3$.
   在这份试卷中，$\lfloor x \rfloor$ 表示小于或等于 x 的最大整数。
   例如：$\lfloor 2 \rfloor = 2$，$\lfloor -2 \rfloor = -2$，$\lfloor 2.6 \rfloor = 2$，$\lfloor -2.6 \rfloor = -3$。

Question M-01 [4 points]

In the decimal expansion of $\frac{3}{13}$, what is the 2020^{th} digit after the decimal point?

将 $\frac{3}{13}$ 以小数形式表示，求在小数点后第 2020 位的数字。

Question M-02 [4 points]

If a convex polygon has 1000 sides, what is the minimum number of obtuse angles that it has? [Each internal angle of a convex polygon is less than $180°$.]

如果一个凸多边形有 1000 个边，则它至少有几个内角是钝角？
[一个凸多边形的每个内角都必须小于 180°。]

Question M-03 [4 points]

If the first 10 terms of the sequence of odd positive integers $1, 3, 5, 7, \ldots$ are merged, we obtain the 15-digit number 135791113151719. If the first n terms of the sequence are merged, we obtain a 2017-digit number. Find the value of n.

将正奇数数列 $1, 3, 5, 7, \ldots$ 的前 10 项连在一起，就得到 135791113151719 这个 15 位数。如果将前 n 项连在一起会得到一个 2017 位数，求 n 的值。

Question M-04 [4 points]

Find the last three digits of $\left(2020^{2020} - 1\right)^{2020}$.

求 $\left(2020^{2020} - 1\right)^{2020}$ 的最后三位数。

Question M-05 [4 points]

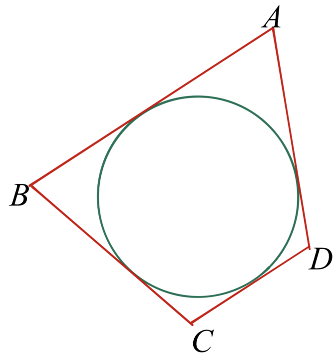
The hypotenuse of a right-angled triangle has length 53, and the lengths of the other two sides differ by 1. Find the area of the triangle.

一直角三角形的斜边长 53，另外两个边的长相差 1。求此三角形的面积。

Question M-06 [4 points]

In the figure shown below, the lines AB, BC, CD and DA are tangents to the circle. If $AB = 101$, $BC = 97$, $CD = 67$, find AD.

下图中，直线 AB，BC，CD 及 DA 都与圆相切。若 $AB = 101$，$BC = 97$，$CD = 67$，求 AD。



Question M-07 [4 points]

Given that n is a positive integer. If there are integers a and b such that $n = 2109a + 5928b$, find the smallest possible value of n.

已知 n 是一正整数且存在整数 a 和 b 使得 $n = 2109a + 5928b$。求 n 的最小可能值。

Question M-08 [4 points]

Given that when the polynomial $f(x)$ is divided by $2x^2 - 191x - 1111$, the remainder is $2x + 50$. Find the remainder when $f(x)$ is divided by $x - 101$.

已知多项式 $f(x)$ 除以 $2x^2 - 191x - 1111$ 得余项 $2x + 50$，求 $f(x)$ 除以 $x - 101$ 的余数。

Question M-09 [4 points]

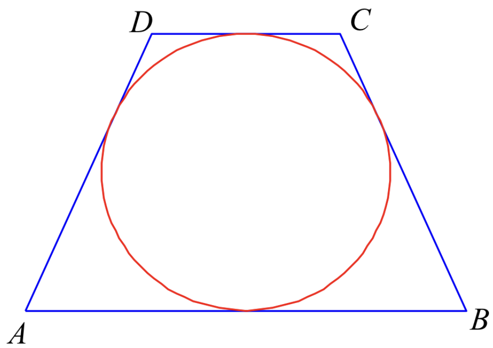
Given a is an integer such that the minimum value of $f(x) = 3x^2 + ax + 10000$ is 253, find the value of a.

已知 a 是一整数且 $f(x) = 3x^2 + ax + 10000$ 的最小值是 253，求 a 的值。

Question M-10 [4 points]

As shown in the figure below, a circle is inscribed in the trapezium $ABCD$. Given that $AD = BC = 65$, and the area of the circle is 784π. Find the length of AB.

如下图所示，一圆内切于梯形 $ABCD$ 中。已知 $AD = BC = 65$ 且圆的面积为 784π，求 AB 的长。



Question M-11 [5 points]

A line separates the plane into 2 regions. Now the plane is separated into N regions by 67 lines, no two of which are parallel. Find the smallest possible value of N.

一条直线将平面分成两个区域。现平面上有 67 条直线，没有两条直线互相平行。若这些直线将平面分成 N 个区域，求 N 的最小可能值。

Question M-12 [5 points]

How many ways are there to divide twelve students into two groups of six each?

有多少种方法可以将 12 位学生分成两组，每组 6 人?

Question M-13 [5 points]

Given that $a_1, a_2, a_3, \ldots$ is a sequence such that $a_n - a_{n-1} = n$ for all $n \geq 2$. If the 40th term of the sequence is 957, find the 20th term of the sequence.

已知 $a_1, a_2, a_3, \ldots$ 是一数列，当 $n \geq 2$ 时，$a_n - a_{n-1} = n$。若此数列的第 40 项是 957，求此数列的第 20 项。

Question M-14 [5 points]

The 2020 students in a school are divided into 7 groups to compete in a game. No two groups have the same number of students. What is the minimum number of students in the largest group?

一间学校有 2020 位学生。如果将这些学生分成 7 组来进行运动会，没有两组有相同的人数，那么人数最多的组至少有几个人?

Question M-15 [5 points]

If k is a positive integer and both the roots of the equation $x^2 - kx + 2020 = 0$ are integers, find the smallest possible value of k.

若 k 是正整数且方程式 $x^2 - kx + 2020 = 0$ 的两个根都是整数，求 k 的最小可能值。

Question M-16 [5 points]

Given that a and b are positive numbers such that $2\sqrt{a}\left(\sqrt{a}+50\sqrt{b}\right)=101\sqrt{b}\left(\sqrt{a}+201\sqrt{b}\right)$. Find

$$\frac{14a-9\sqrt{ab}+b}{a-10000b}.$$

已知 a 与 b 是正数且 $2\sqrt{a}\left(\sqrt{a}+50\sqrt{b}\right)=101\sqrt{b}\left(\sqrt{a}+201\sqrt{b}\right)$。求

$$\frac{14a-9\sqrt{ab}+b}{a-10000b}.$$

Question M-17 [5 points]

A school has four clubs whose members are students in this school. Each club has 177 members. Every two clubs have 52 common members. Every three clubs have 23 common members. There are exactly 11 students that join all four clubs. How many students join exactly one club?

已知一间学校有 4 个学会，每个学会的会员都是该校的学生。每个学会有 177 位会员，每两个学会有 52 位共同会员，每三个学会有 23 位共同会员，恰有 11 位学生是这四个学会的共同会员。只参加一个学会的学生有几人?

Question M-18 [5 points]

Let P be the product of all the solutions of the equation

$$\sqrt{x^2+2x+700}-\sqrt{x^2+2x-99}=17.$$

Find the absolute value of P.

设 P 是方程式

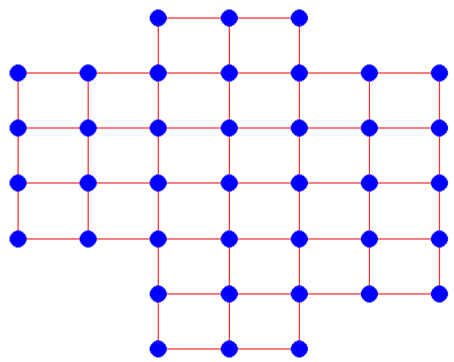
$$\sqrt{x^2+2x+700}-\sqrt{x^2+2x-99}=17$$

所有的解的乘积，求 P 的绝对值。

Question M-19 [5 points]

The figure below shows a network of 39 computers connected by 64 cables. If computer A is connected to computer B by a cable, and computer B is connected to computer C by a cable, then computer A and computer C are connected, even without a cable between them. At most how many of the cables can be removed for the computers to stay connected?

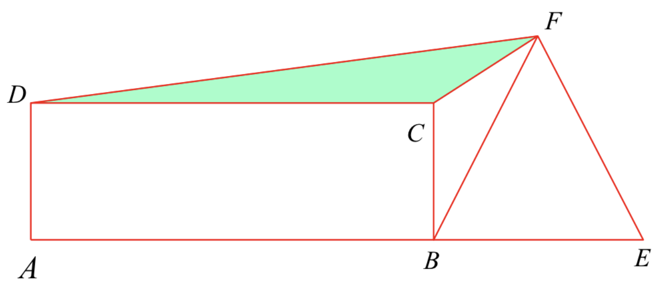
下图显示 39 台电脑被 64 条线路连接起来。如果有线路连接电脑 A 和电脑 B，也有线路连接电脑 B 和电脑 C，则电脑 A 和电脑 C 是互相连通的，即便他们之间没有线路连接。如果要这些电脑之间都相互连通，最多有几条线路可以被拔除？



Question M-20 [5 points]

In the figure shown below, ABE is a straight line. $ABCD$ is a rectangle with $AB = 86$ and $AD = 38$. ΔBEF is an equilateral triangle. ΔBCF is an isosceles triangle. Find the area of ΔCDF.

下图中，ABE 是一直线。$ABCD$ 是长方形，$AB = 86$，$AD = 38$。ΔBEF 是等边三角形，ΔBCF 是一等腰三角形。求 ΔCDF 的面积。



Question M-21 [6 points]

Given that a is a positive constant. When x is a positive number, the minimum value of

$$x^6 + \frac{a^4}{x^6} + 14x^3 + 14\frac{a^2}{x^3}$$

is 702. Find the value of a^2.

已知 a 是正的常数。当 x 是正数时，

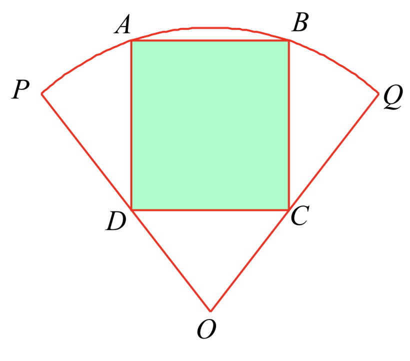
$$x^6 + \frac{a^4}{x^6} + 14x^3 + 14\frac{a^2}{x^3}$$

的最小可能值是 702，求 a^2 的值。

Question M-22 [6 points]

In the figure shown below, OPQ is a sector with $\angle POQ = 60°$. $ABCD$ is a square. If the area of the sector OPQ is 144π, and the area of the square $ABCD$ is S, find the largest integer less than S.

下图中，OPQ 是一扇形，$\angle POQ = 60°$。$ABCD$ 是正方形。已知扇形 OPQ 的面积是 144π，正方形 $ABCD$ 的面积为 S，求小于 S 的最大整数。



Question M-23 [6 points]

If there are k distinct positive integers with sum smaller than 202020, find the largest possible value of k.

如果有 k 个不同的正整数，其和小于 202020，求 k 的最大可能值。

Question M-24 [6 points]

How many pairs of integers (x, y) satisfy the inequality $23|x| + 7|y| \leq 217$?

有多少对整数 (x, y) 满足不等式 $23|x| + 7|y| \leq 217$?

Question M-25 [6 points]

Let $S = \{20x + 101y \,|\, x, y \text{ are nonnegative integers}\}$. Among the positive integers less than 10000, how many of them are not in the set S?

设 $S = \{20x + 101y \,|\, x, y \text{是非负的整数}\}$。在小于 10000 的正整数中，有多少个不在 S 中?

Question M-26 [8 points]

Let S be the set of positive integers less than 236. Given that any n-element subset of S contains two elements that are not relatively prime, find the smallest possible value of n.

设 S 是所有小于 236 的正整数的集合。已知 S 的任一个含有 n 个元素的子集合一定会有两个不互质的元素。求 n 的最小可能值。

Question M-27 [8 points]

Let

$$S = \{n \mid n \text{ is a positive integer less than 100000, the product of digits of } n \text{ is } 120\}$$

Find the number of elements in S.

设

$$S = \{n \mid n \text{ 是小于 100000 的正整数且 } n \text{ 的各位数字的乘积是 } 120\}$$

求 S 中元素的个数。

Question M-28 [8 points]

Let $S = \{1, 3, 5, \ldots, 35\}$ be the set containing all positive odd integers less than 36. For a set A that is a subset of S, define $f(A)$ to be the sum of the elements in A. If $f(A) = 36$, we say that A is perfect. Among all the subsets of S, how many of them are perfect?

设 $S = \{1, 3, 5, \ldots, 35\}$ 为小于 36 的正奇数集合。对于 S 的子集合 A，定义 $f(A)$ 为 A 中所有元素的和。若 $f(A) = 36$，A 就被称为完美的集合。S 的子集中，有几个是完美的?

Question M-29 [8 points]

Let $P = 1 \times 3 \times \ldots \times 999$ be the product of the odd positive integers less than 1000. Find the largest integer k such that 3^k divides P.

设 $P = 1 \times 3 \times \ldots \times 999$ 是小于 1000 的正奇数的乘积。求最大的整数 k 使得 3^k 可以整除 P。

Question M-30 [8 points]

Given that a, b, x and y are real numbers such that

$$\begin{aligned} ax - by &= 16 \\ a^2x^3 - b^2y^3 &= 23 \\ a^3x^5 - b^3y^5 &= 33 \\ a^4x^7 - b^4y^7 &= 48 \\ a^5x^9 - b^5y^9 &= K \end{aligned}$$

Find K.

已知 a，b，x 及 y 是实数且

$$\begin{aligned} ax - by &= 16 \\ a^2x^3 - b^2y^3 &= 23 \\ a^3x^5 - b^3y^5 &= 33 \\ a^4x^7 - b^4y^7 &= 48 \\ a^5x^9 - b^5y^9 &= K \end{aligned}$$

求 K。