Question M-01 [5 points]

If a, b and c are positive integers such that $a + \cfrac{1}{b + \cfrac{1}{c}} = \frac{68}{13}$, find the value of c.

若 a，b 及 c 是正整数使得 $a + \cfrac{1}{b + \cfrac{1}{c}} = \frac{68}{13}$，求 c 的值。

Answer: [3]

Solutions:

Since $0 < \cfrac{1}{b + \cfrac{1}{c}} < 1$, we find that $a = 5$.

$$b + \frac{1}{c} = \frac{13}{3}$$

Therefore, $b = 4$ and $c = 3$.

Question M-02 [5 points]

Given that the positive integer N is divisible by all the prime numbers that divide 2016, find the smallest possible value of N.

已知正整数 N 能被所有能整除 2016 的质数整除，求 N 的最小可能值。

Answer: [42]

Solutions:

$$2016 = 2^5 \times 3^2 \times 7$$

The smallest possible value of N is $2 \times 3 \times 7 = 42$.

Question M-03 [5 points]

In a mathematics competition, the ratio of the points scored by Ailan to the points scored by Beiling is $4 : 3$, the ratio of the points scored by Beiling to the points scored by Dingdang is also $4 : 3$. If the total points scored by Ailan, Beiling and Dingdang is 259, find the points scored by Dingdang.

在一次数学竞赛中，爱兰所得的分数与贝凌所得的分数之比是$4:3$，贝凌所得的分数与叮当所得的分数之比也是$4:3$。若她们三人所得的分数之和是259，求叮当的分数。

Answer: [63]

Solutions:

The points scored by Ailan, Beiling and Dingdang are in the ratio $16 : 12 : 9$.

Hence, the points scored by Dingdang is

$$259 \times \frac{9}{16+12+9} = 63.$$

Question M-04 [5 points]

In the figure below, AB is the diameter of the circle, P, Q, R, S are points on the circle so that

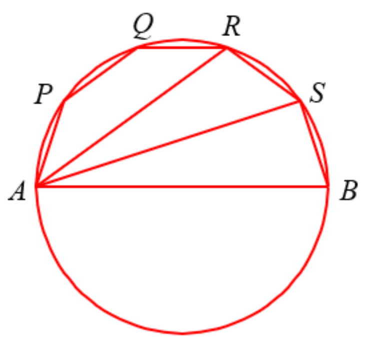
$$AP = PQ = QR = RS = SB.$$

If $\angle RAS = x°$, find x.

下图中，AB 是圆的直径，P，Q，R，S 是圆上的四点使得

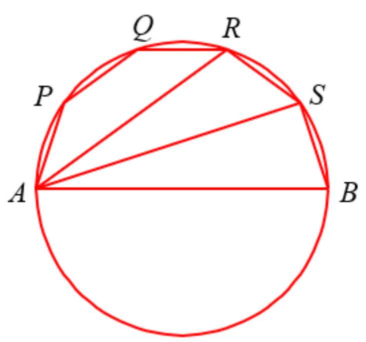
$$AP = PQ = QR = RS = SB$$

若$\angle RAS = x°$，求 x。



Answer: [18]

Solutions:



Let O be the center of the circle.

$$\angle AOP = \angle POQ = \angle QOR = \angle ROS = \angle SOB = \frac{180°}{5} = 36°.$$

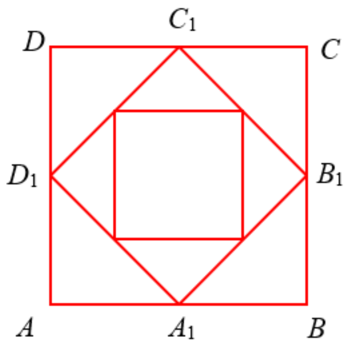
Therefore,

$$\angle RAS = \frac{1}{2}\angle ROS = 18°.$$

Question M-05 [5 points]

As shown in the figure below, $ABCD$ is a square. The midpoints of the sides of $ABCD$ are joined to form the square $A_1B_1C_1D_1$, the midpoints of the sides of $A_1B_1C_1D_1$ are joined to form the square $A_2B_2C_2D_2$. This process continues to form the squares $A_nB_nC_nD_n$, $n \geq 1$. If the area of the square $ABCD$ is 2048, find the area of the square $A_6B_6C_6D_6$.

如下图所示，$ABCD$是一个正方形。将$ABCD$ 各边的中点连起来就得到正方形$A_1B_1C_1D_1$。将$A_1B_1C_1D_1$各边的中点连起来就得到正方形$A_2B_2C_2D_2$。不断重复这过程以得到正方形$A_nB_nC_nD_n$，$n \geq 1$。若正方形$ABCD$的面积是2048，求正方形$A_6B_6C_6D_6$的面积。



Answer: [32]

Solutions:

The area of $A_nB_nC_nD_n$ is half of the area of $A_{n-1}B_{n-1}C_{n-1}D_{n-1}$. Hence, the area of $A_6B_6C_6D_6$ is

$$2048 \times \frac{1}{2^6} = 32.$$

Question M-06 [5 points]

Let $N = \sqrt{2021^2 - 2020^2}$. Find the integer closest to N.

设 $N = \sqrt{2021^2 - 2020^2}$。求最靠近 N 的整数。

Answer: [64]

Solutions:

$$N = 63.57$$

Hence, the integer closest to N is 64.

Question M-07 [5 points]

The faces of a cube with side length 10 is coloured red, and then the cube is dissected into 1000 small cubes with side lengths 1. Among the small cubes, how many of them carry at least a red face?

将一个边长为 10 的立方体的表面涂上红色后，分解为 1000 个边长为 1 的小立方体。在这些小立方体中，有几个小立方体至少有一面被涂上红色?

Answer: [488]

Solutions:

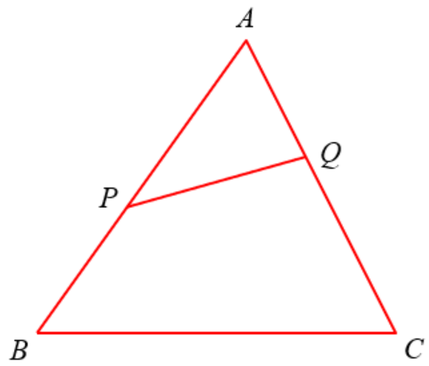
Among the 1000 small cubes, $8^3 = 512$ of them which are in the inner layer do not have any face coloured red. Hence, the number of small cubes that carry at least a red face is

$$1000 - 512 = 488.$$

Question M-08 [5 points]

In the figure below, $AP : PB = 9 : 7$, $AQ : QC = 4 : 5$. Given that the areas of ΔABC and ΔAPQ are S_1 and S_2 respectively. If S_1 and S_2 are integers, find the smallest possible value of $S_1 + S_2$.

下图中，$AP : PB = 9 : 7$，$AQ : QC = 4 : 5$。已知 ΔABC 及ΔAPQ 的面积分别为 S_1 及 S_2，且 S_1 及 S_2都是整数，求 $S_1 + S_2$ 的最小可能值。



Answer: [5]

Solutions:

$$\frac{S_1}{S_2} = \frac{\frac{1}{2} \times AP \times AQ \times \sin A}{\frac{1}{2} \times AB \times AC \times \sin A} = \frac{9}{16} \times \frac{4}{9} = \frac{1}{4}.$$

When $S_1 + S_2$ is smallest, $S_1 = 1$ and $S_2 = 4$. This gives the smallest possible value of $S_1 + S_2$ to be 5.

Question M-09 [5 points]

If x is a positive number such that $\sqrt{x+582}+\sqrt{x+1}=83$, find the value of

$$\sqrt{x+582}-\sqrt{x+1}.$$

已知正数 x 满足方程式 $\sqrt{x+582}+\sqrt{x+1}=83$，求 $\sqrt{x+582}-\sqrt{x+1}$ 的值。

Answer: [7]

Solutions:

$$\left(\sqrt{x+582}+\sqrt{x+1}\right)\left(\sqrt{x+582}-\sqrt{x+1}\right)=581$$
$$\left(\sqrt{x+582}-\sqrt{x+1}\right)=\frac{581}{83}=7$$

Question M-10 [5 points]

A cube and a sphere have the same surface area. If the volume of the cube is V_1, the volume enclosed by the sphere is V_2, find $\pi\left(\frac{V_2}{V_1}\right)^2$.

一个立方体和一粒球有相同的表面积。如果立方体的体积是V_1，球的体积是V_2，求$\pi\left(\frac{V_2}{V_1}\right)^2$。

Answer: [6]

Solutions:

Let the side length of the cube be a, and the radius of the sphere be r. Then

$$6a^2 = 4\pi r^2$$

$$\left(\frac{a}{r}\right)^2 = \frac{2\pi}{3}$$

$$\pi\left(\frac{V_2}{V_1}\right)^2 = \pi\left(\frac{\frac{4}{3}\pi r^3}{a^3}\right)^2 = \frac{16}{9}\pi^3 \times \frac{27}{8\pi^3} = 6$$

Question M-11 [5 points]

N students are sitting in a classroom that has 100 seats. A teacher asks some questions and find out that $\frac{1}{2}$ of the students are females; $\frac{1}{3}$ of the students wear glasses; $\frac{1}{4}$ of the students do not have calculators; $\frac{1}{7}$ of the students do not speak Chinese. Find the value of N.

有 N 位学生坐在一间有100个座位的教室里。老师问了一些问题后，发现有 $\frac{1}{2}$ 的学生是女生，$\frac{1}{3}$ 的学生有戴眼镜，$\frac{1}{4}$ 的学生没有计算机，$\frac{1}{7}$ 的学生不会讲华语。求 N 的值。

Answer: [84]

Solutions:

N must be a common multiple of 2, 3, 4, 7 that is less than 100. Hence, N is 84.

Question M-12 [5 points]

Among the positive integers less than 1000, how many of them are divisible by 6 but not by 15?

小于 1000 的正整数中，有多少个可以被 6 整除，但不能被 15 整除？

Answer: [133]

Solutions:

Integers that are divisible by 6 and by 15 are those that are divisible by 30.

Among the positive integers less than 1000,

166 of them is divisible by 6.

33 of them is divisible by 30.

Hence, the number of integers that are divisible by 6 but not by 15 is $166 - 33 = 133$.

Question M-13 [5 points]

Find the sum of the digits in the number $10^{101} - 101$.

求 $10^{101} - 101$ 这个数的各位数字之和。

Answer: [908]

Solutions:

The number $10^{101} - 101$ contains 101 digits, each of the first 98 digits is 9, the last three digits are 899. Hence, the sum of the digits is $100 \times 9 + 8 = 908$.

Question M-14 [5 points]

If x and y are real numbers such that $x + 2y = 5$ and $3x^2 + 7xy + 2y^2 = 100$, find the value of $4x^2 + y^2$.

若 x 及 y 是实数且 $x + 2y = 5$，$3x^2 + 7xy + 2y^2 = 100$，求 $4x^2 + y^2$ 的值。

Answer: [197]

Solutions:

$$3x^2 + 7xy + 2y^2 = 100$$
$$(3x + y)(x + 2y) = 100$$
$$3x + y = 20$$

This gives $x = 7$, $y = -1$,

$$4x^2 + y^2 = 197.$$

Question M-15 [5 points]

Lingling travelled on a bicycle from town A to town D, passes through town B and town C. From town A to town B, she travelled at a uniform speed of 12 km/h for 30 minutes. From town C to town D, she travelled at a uniform speed of 9 km/h for 1 hour. Given that she travelled at a uniform speed of v km/h from town B to town C, and this speed is also the average speed of the whole journey, find the value of $6v$.

玲玲骑脚车从小镇 A 到小镇 D，途中经过小镇 B 及 C。从小镇 A 到小镇 B，她以每小时 12 公里的均匀速度骑了 30 分钟。从小镇 C 到小镇 D，她以每小时 9 公里的均匀速度骑了 1 小时。已知玲玲是以每小时 v 公里的均匀速度由小镇 B 骑到小镇 C，而这也是她全程的平均速度，求 $6v$ 的值。

Answer: [60]

Solutions:

Assume that Lingling spent h hours on the journey from B to C. Then

$$\frac{12 \times \frac{1}{2} + 9 \times 1 + vh}{\frac{1}{2} + 1 + h} = v$$

$$15 + vh = \frac{3}{2}v + vh$$

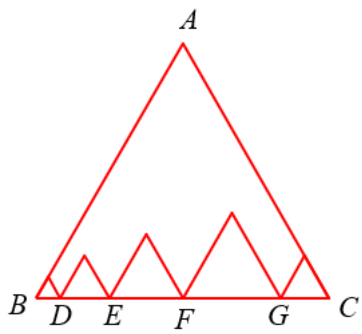
$$v = 10$$

$$6v = 60$$

Question M-16 [5 points]

The figure below shows a large equilateral triangle ΔABC with base BC, and five small equilateral triangles with bases BD, DE, EF, FG and GC respectively. If the perimeter of ΔABC is 219, find the sum of the perimeters of the six equilateral triangles.

下图所示是一个以 BC 为底的大等边三角形 ΔABC，以及五个分别以 BD，DE，EF，FG 及 GC 为底的小等边三角形。若 ΔABC 的周长是 219，求六个等边三角形的周长之和。



Answer: [438]

Solutions:

The sum of the perimeters of the six equilateral triangles is

$$3 \times (BC + BD + DE + EF + FG + GC) = 6 \times BC = 2 \times 219 = 438$$

Question M-17 [5 points]

Given that α and β are the two roots of the equation $x^2 - 119x + 17 = 0$, find the value of $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$.

已知 α 及 β 是方程式 $x^2 - 119x + 17 = 0$ 的两个根，求 $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ 的值。

Answer: [831]

Solutions:

$$\begin{aligned}\frac{\alpha}{\beta} + \frac{\beta}{\alpha} &= \frac{\alpha^2 + \beta^2}{\alpha\beta} \\ &= \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta} \\ &= \frac{119^2 - 2 \times 17}{17} \\ &= 831\end{aligned}$$

Question M-18 [5 points]

Given that x is a real number, find the smallest possible value of

$$|x-1| + |x-2| + \ldots + |x-49| + |x-50|.$$

已知x 是实数，求

$$|x-1| + |x-2| + \ldots + |x-49| + |x-50|$$

的最小可能值。

Answer: [625]

Solutions:

If $x < 1$,

$$\begin{aligned} f(x) =& |x-1| + |x-2| + \ldots + |x-49| + |x-50| \\ =& 1-x+2-x+\ldots+50-x \\ >& 0+1+2+\ldots+48+49 \\ =& \frac{49\times 50}{2} \end{aligned}$$

If $x \geq 50$,

$$\begin{aligned} f(x) =& |x-1| + |x-2| + \ldots + |x-49| + |x-50| \\ =& x-1+x-2+\ldots+x-49+x-50 \\ \geq& 49+48+\ldots+1+0 \\ =& \frac{49\times 50}{2} \end{aligned}$$

For $1 \leq n \leq 49$, if $n \leq x < n+1$, then

$$\begin{aligned} f(x) =& |x-1| + |x-2| + \ldots + |x-49| + |x-50| \\ =& x-1+x-2+\ldots+x-n+(n+1-x)+\ldots+49-x+50-x \\ =& (2n-50)x+(1+2+\ldots+50)-2\,(1+2+\ldots+n) \\ =& (2n-50)x+\frac{50\times 51}{2}-n(n+1) \end{aligned}$$

Therefore, if $n \leq 24$, $f(x)$ is decreasing in the interval $n \leq n < n+1$.

If $n \geq 26$, $f(x)$ is increasing in the interval $n \leq x < n+1$.

$f(x)$ has constant value when $25 \le x \le 26$.

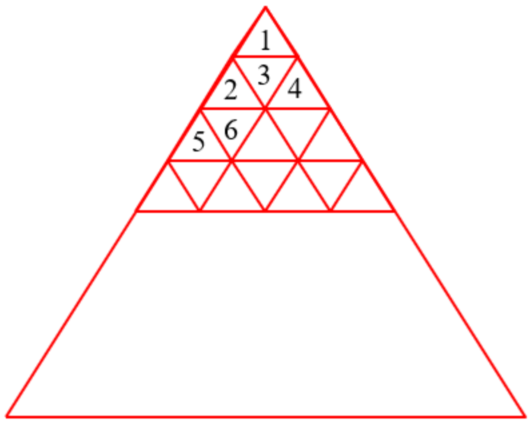
This implies that $f(x)$ has minimum value

$$\frac{50 \times 51}{2} - 25 \times 26 = 625.$$

Question M-19 [5 points]

As shown in the figure below, there are multiple rows of congruent triangles inside a large triangle. The positive integers are filled into the small triangles from left to right, row by row, starting from the first row on the top. If N is the 2021^{th} integer from the left on the 2021^{th} row, find the last three digits of N.

如下图所示，一个大三角形中有很多行小的全等三角形。将正整数填入小三角形中，由上面第一行开始，每行按由左到右的顺序填。若第 2021 行的第 2021 个数是 N，求 N 的最后三位数。



Answer: [421]

Solutions:

The last integer at the end of the 2020^{th} row is

$$1 + 3 + 5 + \ldots + (2 \times 2020 - 1) = 2020^2$$

Therefore,

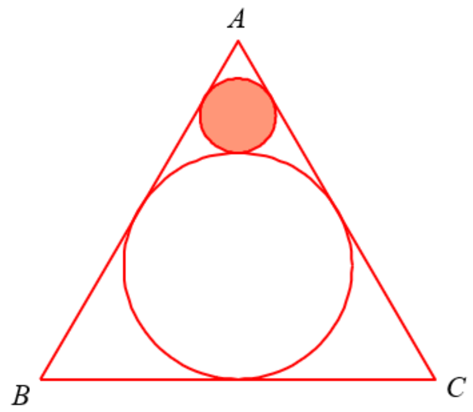
$$N = 2020^2 + 2021 = 4082421$$

The last three digits of N are 421.

Question M-20 [5 points]

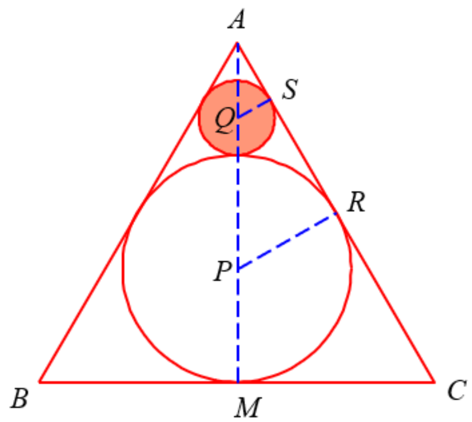
In the figure below, ΔABC is an equilateral triangle. The larger circle is inscribed in the triangle, and the smaller circle is tangent to AB, AC and the larger circle. If the area of the larger circle is 2079, find the area of the smaller circle.

下图中，ΔABC 是等边三角形。大圆是三角形的内切圆，而小圆分别与 AB，AC 及大圆相切。若大圆的面积为 2079，求小圆的面积。



Answer: [231]

Solutions:



Let the radius of the smaller circle be $QS = r$, the radius of the larger circle be $PR = a$. Then

$$PQ = r + a,$$

$$AQ = 2r,$$

$$CM = CR = \sqrt{3}a,$$

$$AM = 3a = 3r + 2a.$$

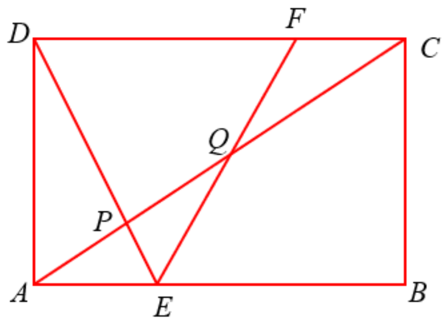
Hence, we find that $a = 3r$.

Therefore, the area of the smaller circle is $\frac{1}{9}$ of the area of the larger circle, which is 231.

Question M-21 [6 points]

As shown in the figure below, $ABCD$ is a rectangle. E is a point on AB with $AE : EB = 1 : 2$, F is a point on CD with $CF : FD = 1 : 3$. The line DE intersects the diagonal AC at the point P, and the line EF intersects AC at the point Q. Find $840 \times \frac{PQ}{AC}$.

下图中，$ABCD$是一个长方形。E是AB上的一点使得$AE : EB = 1 : 2$，F是CD上的一点使得$CF : FD = 1 : 3$。直线DE与对角线AC 相交于点P，而直线EF 与AC 相交于点Q。求 $840 \times \frac{PQ}{AC}$。



Answer: [270]

Solutions:

$$AE = \frac{1}{3}AB = \frac{1}{3}CD$$
$$CF = \frac{1}{4}CD = \frac{1}{4}AB$$
$$\frac{AP}{PC} = \frac{AE}{CD} = \frac{1}{3}\ AP = \frac{1}{4}AC$$
$$\frac{CQ}{AQ} = \frac{CF}{AE} = \frac{3}{4}$$
$$CQ = \frac{3}{7}AC$$
$$PQ = \left(1 - \frac{1}{4} - \frac{3}{7}\right)AC$$
$$840 \times \frac{PQ}{AC} = 270$$

Question M-22 [6 points]

If n is a positive integer such that $\dfrac{140}{n-1} - \dfrac{140}{n+1}$ is also an integer, find the largest possible value of n.

若 n 是一正整数使得 $\dfrac{140}{n-1} - \dfrac{140}{n+1}$ 也是一个整数，求 n 的最大可能值。

Answer: [6]

Solutions:

$\dfrac{140}{n-1} - \dfrac{140}{n+1} = \dfrac{280}{n^2-1}$ is an integer, then $280 = 2^3 \times 5 \times 7$ must be divisible by $n^2 - 1$. The factors of 280 in descending order are 280, 140, 70, 56, 40, 35, ..., we find that 281, 141, 71, 57, 41 are not square numbers but 36 is. Hence, the largest possible value of n is 6.

Question M-23 [6 points]

Given that each interior angle, measured in degrees, of a convex 17-gon is a positive integer, and α, β, γ are three of its interior angles. If $\alpha+\beta+\gamma=x^\circ$, find the minimum possible value of x.

已知一凸十七边形的每一个内角的度数都是正整数，且α，β，γ 是它的其中三个内角的度数。若 $\alpha+\beta+\gamma=x^\circ$，求 x 的最小可能值。

Answer: [194]

Solutions:

The sum of the interior angles is $15\times 180^\circ$. Every interior angle must be less than 180°, and hence must be at most 179°. Therefore,

$$\alpha+\beta+\gamma \geq 15\times 180^\circ - 14\times 179^\circ = 194^\circ.$$

Question M-24 [6 points]

Given that a, b, c are real numbers such that $a + 2b + 3c = 98$, find the minimum value of $a^2 + b^2 + c^2$.

已知 a，b，c 是实数且 $a + 2b + 3c = 98$，求 $a^2 + b^2 + c^2$ 的最小值。

Answer: [686]

Solutions:

Consider the quadratic polynomial

$$\begin{aligned} f(t) &= (t + a)^2 + (2t + b)^2 + (3t + c)^2 \\ &= 14t^2 + 2t(a + 2b + 3c) + (a^2 + b^2 + c^2) \\ &= 14t^2 + 196t + (a^2 + b^2 + c^2) \\ &= 14(t + 7)^2 + (a^2 + b^2 + c^2) - 686 \end{aligned}$$

By definition, $f(t) \geq 0$ for all t. Hence, $a^2 + b^2 + c^2 - 686 \geq 0$.

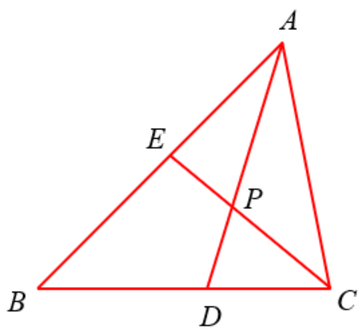
Equality can appear when $t = -7$, $a = 7$, $b = 14$, $c = 21$.

Hence, the minimum value of $a^2 + b^2 + c^2$ is 686.

Question M-25 [6 points]

In the figure below, AD bisects $\angle BAC$, CE bisects $\angle ACB$. AD and CE intersect at the point P. Given that $AB = 12$, $BC = 9$, $AC = 8$. If $AP : PD = m : n$, where m and n are relatively prime positive integers, find the value of $m + n$.

下图中，AD 平分 $\angle BAC$，CE 平分 $\angle ACB$，AD 与 CE 相交于点 P。已知 $AB = 12$，$BC = 9$， $AC = 8$。若 m 及 n 是两个互质的正整数使得 $AP : PD = m : n$，求 $m + n$ 的值。



Answer: [29]

Solutions:

$$\frac{BD}{DC} = \frac{AB}{AC} = \frac{3}{2}$$

$$CD = \frac{2}{5} \times 9 = \frac{18}{5}$$

$$\frac{AP}{PD} = \frac{AC}{CD} = \frac{8}{\frac{18}{5}} = \frac{20}{9}$$

$$m = 20, \ n = 9$$

$$m + n = 29$$

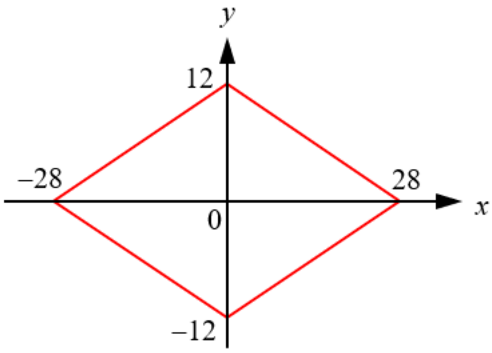
Question M-26 [8 points]

Find the area of the region in the plane defined by the inequalities $|3x - 7y| \leq 84$ and $|3x + 7y| \leq 84$.

求平面上由不等式 $|3x - 7y| \leq 84$ 及 $|3x + 7y| \leq 84$ 所定义的区域的面积。

Answer: [672]

Solutions:



The region is a rhombus as shown in the figure above. Its area is

$$2 \times 28 \times 12 = 672.$$

Question M-27 [8 points]

Given that a, b, c are positive numbers satisfying

$$\begin{aligned} ab + ca &= 323 \\ ab + bc &= 224 \\ bc + ca &= 275 \end{aligned}$$

Find the value of $a^2 + b^2 + c^2$.

已知 a，b，c 是满足方程组

$$\begin{aligned} ab + ca &= 323 \\ ab + bc &= 224 \\ bc + ca &= 275 \end{aligned}$$

的正数，求 $a^2 + b^2 + c^2$ 的值。

Answer: [474]

Solutions:

Adding up the three equations and dividing by 2 give

$$ab + bc + ca = 411$$

This gives

$$\begin{aligned} ab &= 136 = 17 \times 2^3 \\ bc &= 88 = 2^3 \times 11 \\ ca &= 187 = 11 \times 17 \end{aligned}$$

From these we find that $a = 17$, $b = 8$ and $c = 11$.

This gives $a^2 + b^2 + c^2 = 474$.

Question M-28 [8 points]

The number 15 can be written as a sum of consecutive positive integers in three different ways, as shown below.

$$\begin{aligned} 15 =& 7+8 \\ =& 4+5+6 \\ =& 1+2+3+4+5 \end{aligned}$$

In how many ways can the number 2100 be written as a sum of consecutive positive integers?

如下所示，有三种方法可以将15写成连续正整数的和：

$$\begin{aligned} 15 =& 7+8 \\ =& 4+5+6 \\ =& 1+2+3+4+5 \end{aligned}$$

有多少种方法可以将2100写成连续正整数的和?

Answer: [11]

Solutions:

Assume that 2100 is a sum of the n consecutive integers a, $a+1$, $a+2$, $\ldots$, $a+n-1$, where $a \geq 1$. Then

$$n(2a+n-1) = 4200 = 2^3 \times 3 \times 5^2 \times 7$$

Notice that if n is even, $2a+n-1$ is odd and vice versa.

Moreover,

$$2 \leq n < n+1 \leq 2a+n-1.$$

If $4200 = mq$, where m is even and q is odd, then $2^3 \mid m$ and

- 3 divides m but not q, or 3 divides q but not m. There are 2 cases.
- 5^2 divides m and 5 does not divide q, or 5 divides m and q, or 5^2 divides q and 5 does not divide m. There are 3 cases.
- 7 divides m but not q, or 7 divides q but not m. There are 2 cases.

There is a total of $2 \times 3 \times 2 = 12$ cases. Among them, one is $q = 1$.

For each pair of (m, q) obtained above except the one where $(m, q) = (4200, 1)$, let n be the smaller of m and q, and $2a + n - 1$ to be the larger one. Then one can solve for a from the latter and obtain a way to represent 2100 as a sum of consecutive positive integers.

Hence, there are 11 ways.

Question M-29 [8 points]

A triangle is called magic if all its side lengths are integers and its perimeter is equal to 99. If two triangles are congruent, we say that they are the same; otherwise they are distinct. Find the number of distinct magic triangles.

若一个三角形三边的长都是整数且他的周长是99，我们称它为神奇三角形。若两个三角形全等，我们说它们相同；否则就称为相异。求相异的神奇三角形的个数。

Answer: [217]

Solutions:

A triangle is uniquely determined by its side length a, b, c with $a \leq b \leq c$.

Then triangle inequality implies that $a + b > c$.

For a magic triangle,

$$2c < a + b + c = 99$$

but

$$3c \geq a + b + c = 99.$$

Hence, c is an integer between 33 and 49.

When $c = 33$, (a, b) can only be $(33, 33)$. 1 case.

When $c = 34$, (a, b) can only be $(31, 34)$ or $(32, 33)$. 2 cases.

When $c = 35$, (a, b) can only be $(29, 35)$, $(30, 34)$, $(31, 33)$, $(32, 32)$. 4 cases.

When $c = 36$, (a, b) can only be $(27, 36)$, $(28, 35)$, $(29, 34)$, $(30, 33)$, $(31, 32)$. 5 cases.

$$\vdots$$

When $c = 49$, (a, b) can only be $(1, 49)$, $(2, 48)$, $\ldots$, $(25, 25)$. 25 cases.

Hence, the number of noncongruent magic triangles is

$$(1 + 4 + 7 + \ldots + 25) + (2 + 5 + \ldots + 23) = 217$$

Question M-30 [8 points]

We say that a positive integer n is lovely if there exists exactly one integer m satisfying

$$\frac{28}{29} < \frac{n}{n+m} < \frac{29}{30}.$$

Find the number of lovely integers.

如果n 是一个正整数且恰有一个正整数m使得

$$\frac{28}{29} < \frac{n}{n+m} < \frac{29}{30},$$

则我们说n是可爱的整数。求可爱的整数的个数。

Answer: [812]

Solutions:

$$\frac{28}{29} < \frac{n}{n+m} < \frac{29}{30}$$
$$\frac{30}{29} < 1+\frac{m}{n} < \frac{29}{28}$$
$$\frac{1}{29} < \frac{m}{n} < \frac{1}{28}$$
$$28m < n < 29m$$

When $m = 1$, no n satisfies the inequality.

When $m = 2$, n can be from $28 \times 2 + 1$ to $29 \times 2 - 1$.

When $m = 3$, n can be from $28 \times 3 + 1$ to $29 \times 3 - 1$.

When $m = 4$, n can be from $28 \times 4 + 1$ to $29 \times 4 - 1$.

$$\vdots$$

When $m = 28$, n can be from $28 \times 28 + 1$ to $29 \times 28 - 1$.

When $m = 29$, n can be from $28 \times 29 + 1$ to $29 \times 29 - 1$.

When $m = 30$, n can be from $28 \times 30 + 1$ to $29 \times 30 - 1$, notice that $28 \times 30 + 1 = 29 \times 29$.

When $m = 31$, n can be from $28 \times 31 + 1$ to $29 \times 31 - 1$, but

$$28 \times 31 + 1 = 29 \times 30 - 1$$

When $m = 32$, n can be from $28 \times 32 + 1$ to $29 \times 32 - 1$, but

$$28 \times 32 + 1 = 29 \times 31 - 2$$

When $m = 33$, n can be from $28 \times 33 + 1$ to $29 \times 33 - 1$, but

$$28 \times 33 + 1 = 29 \times 32 - 3$$

$$\vdots$$

When $m = 56$, n can be from $28 \times 56 + 1$ to $29 \times 56 - 1$, but

$$28 \times 56 + 1 = 29 \times 55 - 26$$

When $m = 57$, n can be from $28 \times 57 + 1$ to $29 \times 57 - 1$, but

$$28 \times 57 + 1 = 29 \times 56 - 27 = 28 \times 56 + 29$$

If $m \geq 58$, n can be from $28m + 1$ to $29m - 1$, but

$$29(m-1) - m + 30 = 28m + 1$$
$$29(m-1) - 1 = 28m + m - 30$$
$$28(m+1) + 1 = 28m + 29$$
$$28(m+1) + m - 29 = 29m - 1$$

Hence, each of the n from $28m + 1$ to $29m - 1$ has two corresponding m.

Removing the n that has more than one m, we find that the lovely integers are

- $28 \times 2 + 1$ to $29 \times 2 - 1$, there is 1 of them.
- $28 \times 3 + 1$ to $29 \times 3 - 1$, there are 2 of them.
- $28 \times 4 + 1$ to $29 \times 4 - 1$, there are 3 of them.

$$\vdots$$

- $28 \times 29 + 1$ to $29 \times 29 - 1$, there are 28 of them.
- $28 \times 30 + 1$ to $29 \times 30 - 2$, there are 28 of them.
- $28 \times 31 + 2$ to $29 \times 31 - 3$, there are 27 of them.
- $28 \times 32 + 3$ to $29 \times 32 - 4$, there are 26 of them.

$$\vdots$$

- $28 \times 57 + 28$ to $29 \times 57 - 29$, there is 1 of them.

There is a total of

$$2(1 + 2 + \ldots + 28) = 28 \times 29 = 812$$

lovely integers.