厦门大学马来西亚分校
陈景润杯中学数学竞赛

CHEN JINGRUN'S CUP SECONDARY SCHOOL MATHEMATICS COMPETITION

INTERMEDIATE CATEGORY

** 中阶组 **

Date: 1st April 2021 | Time: 10:00 a.m. to 12:30 p.m.

日期：2021 年 4 月 1 日 | 时间：上午 10 时至下午 12 时 30 分

考生须知

Information and Instruction

这是**中阶组**的题目的线下版本。

This is the off-line version of the question paper for the **INTERMEDIATE** category.

负责老师会发给你一个 Microsoft Form 的链接，你必须上线在 Form 上填写答案提交。请务必在正确的 Form 上提交答案。

You need to go online to enter the answers to a Microsoft Form whose link is shared to you by your Teacher in Charge. Make sure you submit the correct form.

Microsoft Form 将在下午 12 点 30 分关闭，请在 12 点 25 分之前就按下"提交"键，以免因为网络问题而无法提交。

The Microsoft Form would be closed at 12:30pm. Please press the "Submit" button before 12:25pm, to avoid failure of submission due to network problems.

请仔细阅读下一页的资讯与指示，再继续作答。

Read through the information and instructions given in next page carefully. Then proceed to answer the questions.

在 Microsoft Form 的最后一页，你必须填写个人资料，再提交表格。请务必填写正确的**考生编号**。考生编号不是学号，也不是身份证号码。它是主办方提供的编号。如果你不清楚自己的考生编号，请询问学校的负责老师。

You are required to enter your personal information in the last page of the Microsoft Form, before you submit the form. Please enter your **candidate ID** correctly. The candidate ID is not your student ID, and is also not your IC number. It is a unique code given by the organizer. If you do not know your candidate ID, please ask the teacher in charge of your school.

一共有 30 题，**每题的答案都是一个整数。**每题在一个页面。请在 Microsoft Form 相应的空格内填入答案，**答案必须是一个数字，不用写单位，不要填入数字以外的字元。**

There are altogether 30 questions. **The answer to each question is an integer.** Each question is on a single page. Please enter your answer in the corresponding empty box in the Microsoft Form. **The answer must be a number. Do not write the units. Do not enter characters other than numbers.**

第 1 至 第 20 题，每题 5 分	Question 1 to Question 20, each question carries 5 points
第 21 至 第 25 题，每题 6 分	Question 21 to Question 25, each question carries 6 points
第 26 至 第 30 题，每题 8 分	Question 26 to Question 30, each question carries 8 points
总分是 170 分	Total is 170 points.

学术诚信

Academic Honesty

参赛者必须独立参与比赛。不可得到其他人的协助，在比赛期间不能以任何方式和任何人互通信息。如果违反这些规则，参赛资格将被取消，而且也会受到学校的严厉纪律处分。

Participants should work independently in this competition. You should not get help from others, and should not communicate with anyone by any means during the competition. If you are found to violate these rules, you would be disqualified and face severe disciplinary action from your school.

关于线上提交答案的温馨提示

Kind Reminder for Submitting Answers to Microsoft Form online

1. "Submit"键在最后一页，在比赛时间结束前，请记得按下该键。
   The "Submit" button is at the last page. Before the end of the exam time, please remember to press the "Submit" button.

2. 有打 " * " 的项目一定要作答，否则无法提交。
   It is compulsory to response to all items with " * ", or else you would not be able to submit.

3. 未按“Submit" 键之前，一旦刷新或关掉浏览器，所有填入的资料将消失。
   If you refresh or close your browser before you press the "Submit" button, all the information that you have filled in will be lost.

4. 请在草稿纸上记录各题的答案，一旦不小心刷新或关掉浏览器，请重新登入 Microsoft Form 再填写一次。
   Please record your answers to each question on a piece of draft paper. If you accidentally refresh or close your browser, login the Microsoft form again to fill in all the answers.

5. 如果重复提交答案，必须提供合理的解释。
   If the form is submitted more than once, you need to give a good reason.

Question M-01 [5 points]

If a, b and c are positive integers such that $a + \cfrac{1}{b + \cfrac{1}{c}} = \dfrac{68}{13}$, find the value of c.

若 a，b 及 c 是正整数使得 $a + \cfrac{1}{b + \cfrac{1}{c}} = \dfrac{68}{13}$，求 c 的值。

Question M-02 [5 points]

Given that the positive integer N is divisible by all the prime numbers that divide 2016, find the smallest possible value of N.

已知正整数 N 能被所有能整除 2016 的质数整除，求 N 的最小可能值。

Question M-03 [5 points]

In a mathematics competition, the ratio of the points scored by Ailan to the points scored by Beiling is $4 : 3$, the ratio of the points scored by Beiling to the points scored by Dingdang is also $4 : 3$. If the total points scored by Ailan, Beiling and Dingdang is 259, find the points scored by Dingdang.

在一次数学竞赛中，爱兰所得的分数与贝凌所得的分数之比是$4 : 3$，贝凌所得的分数与叮当所得的分数之比也是$4 : 3$。若她们三人所得的分数之和是259，求叮当的分数。

Question M-04 [5 points]

In the figure below, AB is the diameter of the circle, P, Q, R, S are points on the circle so that

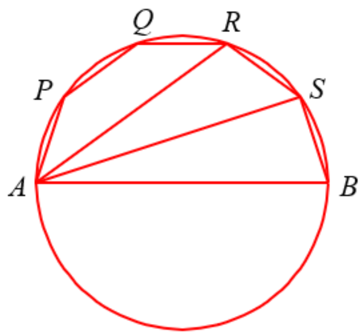
$$AP = PQ = QR = RS = SB.$$

If $\angle RAS = x°$, find x.

下图中，AB 是圆的直径，P，Q，R，S 是圆上的四点使得

$$AP = PQ = QR = RS = SB$$

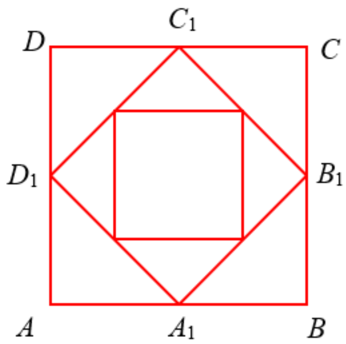
若$\angle RAS = x°$，求 x。



Question M-05 [5 points]

As shown in the figure below, $ABCD$ is a square. The midpoints of the sides of $ABCD$ are joined to form the square $A_1B_1C_1D_1$, the midpoints of the sides of $A_1B_1C_1D_1$ are joined to form the square $A_2B_2C_2D_2$. This process continues to form the squares $A_nB_nC_nD_n$, $n \geq 1$. If the area of the square $ABCD$ is 2048, find the area of the square $A_6B_6C_6D_6$.

如下图所示，$ABCD$是一个正方形。将$ABCD$ 各边的中点连起来就得到正方形$A_1B_1C_1D_1$。将$A_1B_1C_1D_1$各边的中点连起来就得到正方形$A_2B_2C_2D_2$。不断重复这过程以得到正方形$A_nB_nC_nD_n$，$n \geq 1$。若正方形$ABCD$的面积是2048，求正方形$A_6B_6C_6D_6$的面积。



Question M-06 [5 points]

Let $N = \sqrt{2021^2 - 2020^2}$. Find the integer closest to N.

设 $N = \sqrt{2021^2 - 2020^2}$。求最靠近 N 的整数。

Question M-07 [5 points]

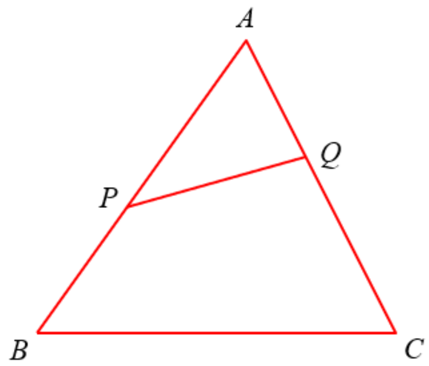
The faces of a cube with side length 10 is coloured red, and then the cube is dissected into 1000 small cubes with side lengths 1. Among the small cubes, how many of them carry at least a red face?

将一个边长为 10 的立方体的表面涂上红色后，分解为 1000 个边长为 1 的小立方体。在这些小立方体中，有几个小立方体至少有一面被涂上红色?

Question M-08 [5 points]

In the figure below, $AP : PB = 9 : 7$, $AQ : QC = 4 : 5$. Given that the areas of ΔABC and ΔAPQ are S_1 and S_2 respectively. If S_1 and S_2 are integers, find the smallest possible value of $S_1 + S_2$.

下图中，$AP : PB = 9 : 7$，$AQ : QC = 4 : 5$。已知 ΔABC 及ΔAPQ 的面积分别为 S_1 及 S_2，且 S_1 及 S_2都是整数，求 $S_1 + S_2$ 的最小可能值。



Question M-09 [5 points]

If x is a positive number such that $\sqrt{x+582}+\sqrt{x+1}=83$, find the value of

$$\sqrt{x+582}-\sqrt{x+1}.$$

已知正数 x 满足方程式 $\sqrt{x+582}+\sqrt{x+1}=83$，求 $\sqrt{x+582}-\sqrt{x+1}$ 的值。

Question M-10 [5 points]

A cube and a sphere have the same surface area. If the volume of the cube is V_1, the volume enclosed by the sphere is V_2, find $\pi\left(\frac{V_2}{V_1}\right)^2$.

一个立方体和一粒球有相同的表面积。如果立方体的体积是V_1，球的体积是V_2，求$\pi\left(\frac{V_2}{V_1}\right)^2$。

Question M-11 [5 points]

N students are sitting in a classroom that has 100 seats. A teacher asks some questions and find out that $\frac{1}{2}$ of the students are females; $\frac{1}{3}$ of the students wear glasses; $\frac{1}{4}$ of the students do not have calculators; $\frac{1}{7}$ of the students do not speak Chinese. Find the value of N.

有 N 位学生坐在一间有100个座位的教室里。老师问了一些问题后，发现有 $\frac{1}{2}$ 的学生是女生，$\frac{1}{3}$ 的学生有戴眼镜，$\frac{1}{4}$ 的学生没有计算机，$\frac{1}{7}$ 的学生不会讲华语。求 N 的值。

Question M-12 [5 points]

Among the positive integers less than 1000, how many of them are divisible by 6 but not by 15?

小于 1000 的正整数中，有多少个可以被 6 整除，但不能被 15 整除？

Question M-13 [5 points]

Find the sum of the digits in the number $10^{101} - 101$.

求 $10^{101} - 101$ 这个数的各位数字之和。

Question M-14 [5 points]

If x and y are real numbers such that $x + 2y = 5$ and $3x^2 + 7xy + 2y^2 = 100$, find the value of $4x^2 + y^2$.

若 x 及 y 是实数且 $x + 2y = 5$，$3x^2 + 7xy + 2y^2 = 100$，求 $4x^2 + y^2$ 的值。

Question M-15 [5 points]

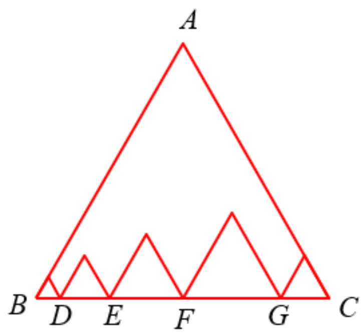
Lingling travelled on a bicycle from town A to town D, passes through town B and town C. From town A to town B, she travelled at a uniform speed of 12 km/h for 30 minutes. From town C to town D, she travelled at a uniform speed of 9 km/h for 1 hour. Given that she travelled at a uniform speed of v km/h from town B to town C, and this speed is also the average speed of the whole journey, find the value of $6v$.

玲玲骑脚车从小镇A到小镇D，途中经过小镇B及C。从小镇A到小镇B，她以每小时12公里的均匀速度骑了30分钟。从小镇C到小镇D，她以每小时9公里的均匀速度骑了1小时。已知玲玲是以每小时v公里的均匀速度由小镇B骑到小镇C，而这也是她全程的平均速度，求$6v$的值。

Question M-16 [5 points]

The figure below shows a large equilateral triangle ΔABC with base BC, and five small equilateral triangles with bases BD, DE, EF, FG and GC respectively. If the perimeter of ΔABC is 219, find the sum of the perimeters of the six equilateral triangles.

下图所示是一个以 BC 为底的大等边三角形 ΔABC，以及五个分别以 BD，DE，EF，FG 及 GC 为底的小等边三角形。若 ΔABC 的周长是 219，求六个等边三角形的周长之和。



Question M-17 [5 points]

Given that α and β are the two roots of the equation $x^2 - 119x + 17 = 0$, find the value of $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$.

已知 α 及 β 是方程式 $x^2 - 119x + 17 = 0$ 的两个根，求 $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ 的值。

Question M-18 [5 points]

Given that x is a real number, find the smallest possible value of

$$|x-1| + |x-2| + \ldots + |x-49| + |x-50|.$$

已知 x 是实数，求

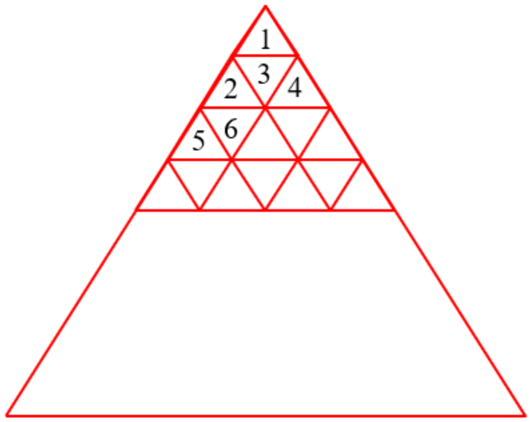
$$|x-1| + |x-2| + \ldots + |x-49| + |x-50|$$

的最小可能值。

Question M-19 [5 points]

As shown in the figure below, there are multiple rows of congruent triangles inside a large triangle. The positive integers are filled into the small triangles from left to right, row by row, starting from the first row on the top. If N is the 2021^{th} integer from the left on the 2021^{th} row, find the last three digits of N.

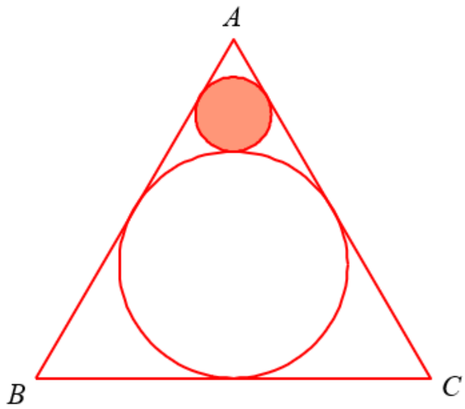
如下图所示，一个大三角形中有很多行小的全等三角形。将正整数填入小三角形中，由上面第一行开始，每行按由左到右的顺序填。若第 2021 行的第 2021 个数是 N，求 N 的最后三位数。



Question M-20 [5 points]

In the figure below, ΔABC is an equilateral triangle. The larger circle is inscribed in the triangle, and the smaller circle is tangent to AB, AC and the larger circle. If the area of the larger circle is 2079, find the area of the smaller circle.

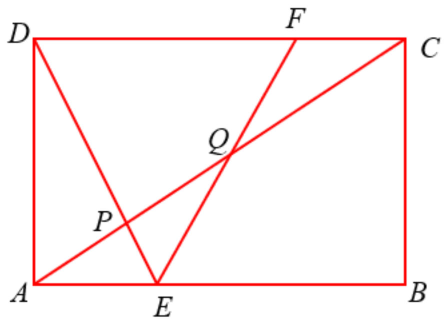
下图中，ΔABC 是等边三角形。大圆是三角形的内切圆，而小圆分别与 AB，AC 及大圆相切。若大圆的面积为 2079，求小圆的面积。



Question M-21 [6 points]

As shown in the figure below, $ABCD$ is a rectangle. E is a point on AB with $AE : EB = 1 : 2$, F is a point on CD with $CF : FD = 1 : 3$. The line DE intersects the diagonal AC at the point P, and the line EF intersects AC at the point Q. Find $840 \times \dfrac{PQ}{AC}$.

下图中，$ABCD$是一个长方形。E是AB上的一点使得$AE : EB = 1 : 2$，F是CD上的一点使得$CF : FD = 1 : 3$。直线DE与对角线AC 相交于点P，而直线EF 与AC 相交于点Q。求 $840 \times \dfrac{PQ}{AC}$。



Question M-22 [6 points]

If n is a positive integer such that $\frac{140}{n-1} - \frac{140}{n+1}$ is also an integer, find the largest possible value of n.

若 n 是一正整数使得 $\frac{140}{n-1} - \frac{140}{n+1}$ 也是一个整数，求 n 的最大可能值。

Question M-23 [6 points]

Given that each interior angle, measured in degrees, of a convex 17-gon is a positive integer, and α, β, γ are three of its interior angles. If $\alpha+\beta+\gamma=x^\circ$, find the minimum possible value of x.

已知一凸十七边形的每一个内角的度数都是正整数，且α，β，γ 是它的其中三个内角的度数。若 $\alpha+\beta+\gamma=x^\circ$，求 x 的最小可能值。

Question M-24 [6 points]

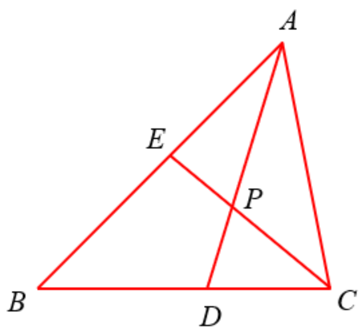
Given that a, b, c are real numbers such that $a + 2b + 3c = 98$, find the minimum value of $a^2 + b^2 + c^2$.

已知 a，b，c 是实数且 $a + 2b + 3c = 98$，求 $a^2 + b^2 + c^2$ 的最小值。

Question M-25 [6 points]

In the figure below, AD bisects $\angle BAC$, CE bisects $\angle ACB$. AD and CE intersect at the point P. Given that $AB = 12$, $BC = 9$, $AC = 8$. If $AP : PD = m : n$, where m and n are relatively prime positive integers, find the value of $m + n$.

下图中，AD 平分 $\angle BAC$，CE 平分 $\angle ACB$，AD 与 CE 相交于点 P。已知 $AB = 12$，$BC = 9$， $AC = 8$。若 m 及 n 是两个互质的正整数使得 $AP : PD = m : n$，求 $m + n$ 的值。



Question M-26 [8 points]

Find the area of the region in the plane defined by the inequalities $|3x - 7y| \leq 84$ and $|3x + 7y| \leq 84$.

求平面上由不等式$|3x - 7y| \leq 84$及$|3x + 7y| \leq 84$所定义的区域的面积。

Question M-27 [8 points]

Given that a, b, c are positive numbers satisfying

$$\begin{aligned} ab + ca &= 323 \\ ab + bc &= 224 \\ bc + ca &= 275 \end{aligned}$$

Find the value of $a^2 + b^2 + c^2$.

已知 a，b，c 是满足方程组

$$\begin{aligned} ab + ca &= 323 \\ ab + bc &= 224 \\ bc + ca &= 275 \end{aligned}$$

的正数，求 $a^2 + b^2 + c^2$ 的值。

Question M-28 [8 points]

The number 15 can be written as a sum of consecutive positive integers in three different ways, as shown below.

$$\begin{aligned} 15 =& 7+8 \\ =& 4+5+6 \\ =& 1+2+3+4+5 \end{aligned}$$

In how many ways can the number 2100 be written as a sum of consecutive positive integers?

如下所示，有三种方法可以将 15 写成连续正整数的和：

$$\begin{aligned} 15 =& 7+8 \\ =& 4+5+6 \\ =& 1+2+3+4+5 \end{aligned}$$

有多少种方法可以将 2100 写成连续正整数的和?

Question M-29 [8 points]

A triangle is called magic if all its side lengths are integers and its perimeter is equal to 99. If two triangles are congruent, we say that they are the same; otherwise they are distinct. Find the number of distinct magic triangles.

若一个三角形三边的长都是整数且他的周长是99，我们称它为神奇三角形。若两个三角形全等，我们说它们相同；否则就称为相异。求相异的神奇三角形的个数。

Question M-30 [8 points]

We say that a positive integer n is lovely if there exists exactly one integer m satisfying

$$\frac{28}{29} < \frac{n}{n+m} < \frac{29}{30}.$$

Find the number of lovely integers.

如果n 是一个正整数且恰有一个正整数m使得

$$\frac{28}{29} < \frac{n}{n+m} < \frac{29}{30},$$

则我们说n是可爱的整数。求可爱的整数的个数。