厦门大学马来西亚分校

陈景润杯中学数学比赛

2017 年第 1 届陈景润杯中学数学比赛

~ 高阶组 ~

日期：2017 年 4 月 1 日

Date: 1st April 2017

时间：下午 2 时至 4 时

Time: 2:00 p.m. to 4:00 p.m.

考生须知

Instructions and Information

1. 本试卷共有 30 题。
   This paper contains 30 questions.
   - 第 1 至第 10 题，选择题，每题 4 分。
     Question 1 to Question 10, multiple choice questions, each question carries 4 marks.
   - 第 11 至第 30 题，问答题，每题请填入一个 −1000 至 1000 之间的整数为答案。
     Question 11 to Question 30, short questions. For each question, please fill in an integer between −1000 and 1000 as answer.
     - 第 11 至第 20 题每题 5 分。
       Question 11 to Question 20, each question carries 5 marks.
     - 第 21 至第 25 题每题 6 分。
       Question 21 to Question 25, each question carries 6 marks.
     - 第 26 至第 30 题每题 8 分。
       Question 26 to Question 30, each question carries 8 marks.
2. 请在答案纸内适当的空格中用 2B 铅笔清楚的写出每题的答案。对于选择题，只需填写 A，B，C，D 或 E 作为答案。每题只能填入一个答案，否则以答错论。
   Please use 2B lead pencils to put your answer to each question in the appropriate space provided on the answer sheet. For a multiple choice question, you only need to put in A, B, C, D or E as answer. You can only give an answer to each question, otherwise it is considered that you answer that question incorrectly.
3. 所有的图形并没有按照比例作图，只作为辅助之用。
   All the diagrams are not drawn to scales. They are intended only as aids.
4. 不许使用计算器，数学工具，手机或其他计算器。
   No calculators, maths stencils, mobile phones or other calculating aids are permitted.
5. 在答案纸上清楚写上姓名，考生编号，学校名称及在学年级。
   Write your name, IC number, school name and year of study clearly on the answer sheet.
6. 在监考老师宣布比赛开始之后，才可以翻开此考卷开始作答。
   You can only open this question booklet to start answering questions after the invigilator announce the beginning of the competition.

~~ 说明 ~~

~~ Notes ~~

在这份试卷中，$\lfloor x \rfloor$表示小于或等于x的最大整数。

例如：$\lfloor 2 \rfloor = 2$，$\lfloor -2 \rfloor = -2$，$\lfloor 2.6 \rfloor = 2$，$\lfloor -2.6 \rfloor = -3$。

In this paper, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

For example, $\lfloor 2 \rfloor = 2$, $\lfloor -2 \rfloor = -2$, $\lfloor 2.6 \rfloor = 2$, $\lfloor -2.6 \rfloor = -3$.

第 1 至第 10 题，选择题，每题 4 分。

Question 1 to Question 10, multiple choice questions, each question carries 4 marks.

1. 惠兰买入一双鞋子放在她的店里卖，后来她按标价折扣 20%卖出这双鞋子，结果没有盈亏。则鞋子的标价比惠兰买入的价钱高______。

   Hui Lan bought a pair of shoes and put them on sale in her shop. Later she sold that pair of shoes with 20% discount off the marked price. She neither got any profit nor suffered any loses. The marked price of the shoes is higher than the price Hui Lan paid for it by______.

   A. 16%　　B. 20%　　C. 24%　　D. 25%　　E. 30%

2. 平行四边形$ABCD$中，$\angle A = 120^\circ$，$AB = 3$，$AD = 5$，若ℓ是平行四边形$ABCD$较短的对角线的长，求ℓ^2。

   In the parallelogram $ABCD$, $\angle A = 120^\circ$, $AB = 3$, $AD = 5$. If ℓ is the length of the shorter diagonal of the parallelogram $ABCD$, find ℓ^2.

   A. 18　　B. 19　　C. 34　　D. 49　　E. 50

3. 若$x = \dfrac{23.2 \times 0.049}{0.0031}$，下列哪一个数最接近$x$？

   If $x = \dfrac{23.2 \times 0.049}{0.0031}$, which of the following number is closest to x?

   A. 3　　B. 33　　C. 333　　D. 3333　　E. 33333

4. 若现在的时间是 $09:00$，2413 小时后是什么时间？

If the time now is $09:00$, what is the time 2413 hours later?

A. $22:00$ B. $10:00$ C. $13:00$ D. $1:00$ E. $21:00$

5. 求 $1+2-3+4+5-6+7+8-9+\ldots+100+101-102$。

Find $1+2-3+4+5-6+7+8-9+\ldots+100+101-102$.

A. 1680 B. 1683 C. 1751 D. 3468 E. 5253

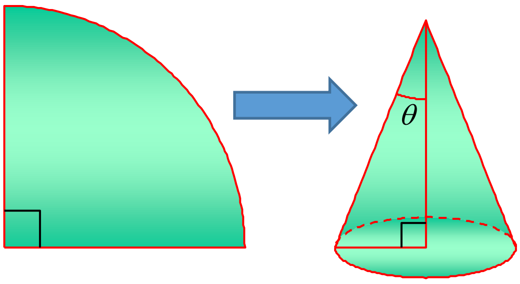
6. 若 $x=1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{\ddots}}}}}}$，求 x。

If $x=1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{\ddots}}}}}}$, find x.

A. $\dfrac{\sqrt{7}+1}{2}$ B. $\dfrac{\sqrt{5}+2}{2}$ C. $\dfrac{\sqrt{5}+1}{2}$ D. $\dfrac{\sqrt{3}+2}{2}$ E. $\dfrac{\sqrt{3}+1}{2}$

7. 如下图所示，一个四分之一圆的卡纸被卷起来粘成一个圆锥形的生日帽。若帽子的对称轴与斜高所形成的角是θ，求$\sin^2\theta$。

As shown in the figure below, a cardboard of the shape of a quarter circle is folded and glued to form a birthday cap. If the angle between the axis of symmetry and the slant height of the cap is θ, find $\sin^2\theta$.



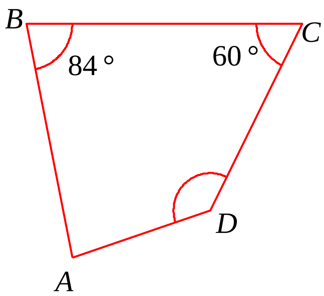
A. $\frac{1}{3}$ B. $\frac{1}{4}$ C. $\frac{1}{6}$ D. $\frac{1}{12}$ E. $\frac{1}{16}$

8. 一家超级市场有 7 个结账柜台，所有柜台都接受现金，但只有其中 4 个柜台接受信用卡。艾雯，冰冰及春兰三人同时买完东西后要付帐，艾雯及冰冰只能用信用卡，而春兰只能用现金。如果她们三人要选择不同的柜台结账，则她们有几种方法可选择结账的柜台?

A supermarket has 7 checkout counters. All these counters accept cash, but only 4 of them accept credit cards. Ai-Wen, Bing-Bing and Chun-Lan finished shopping at the same time, and they want to check out. Ai-Wen and Bing-Bing can only use credit cards, whereas Chun-Lan can only use cash. If they want to choose different counters to check out, how many ways can they choose their checkout counters?

A. 126 B. 120 C. 84 D. 60 E. 30

9. 下图中，$AB=BC=CD$，$\angle B=84°$，$\angle C=60°$，求$\angle D$。
In the figure below, $AB=BC=CD$, $\angle B=84°$, $\angle C=60°$, find $\angle D$.



A. 128° B. 132° C. 134° D. 136° E. 138°

10. 若$500!=1\times2\times3\times\ldots\times500$可被$7^k$整除，求$k$的最大可能值。
If $500!=1\times2\times3\times\ldots\times500$ is divisible by 7^k, find the largest possible value of k.

A. 82 B. 81 C. 71 D. 94 E. 91

第 11 至第 20 题，问答题，每题 5 分。
Question 11 to Question 20, short questions, each question carries 5 marks.

11. 设x，y是两个非零的实数且$x\neq y$。若$x+\frac{9}{x}=y+\frac{9}{y}$，求$xy$。
Let x, y be two nonzero real numbers such that $x\neq y$. If $x+\frac{9}{x}=y+\frac{9}{y}$, find xy.

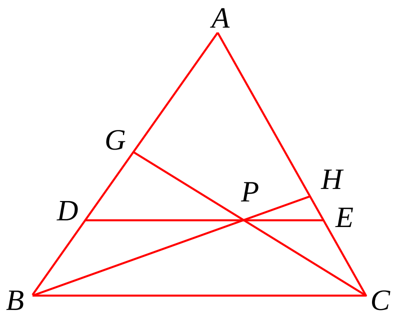
12. 在一次篮球比赛中，彦彰只投2分球或3分球。若他一共投了30次球，有60%的2分球投得中，40%的3分球投得中，问他一共投得几分？
In a basketball game, Yen-Chang only attempted 2-point shots or 3-point shots. If he attempted 30 shots, 60% of the 2-point shots are successful, 40% of the 3-point shots are successful, what is the total points he scored?

13. 有多少个整数 n 满足不等式 $|n-4|+|n-8|\le 24$？

How many integers are there that satisfy the inequality $|n-4|+|n-8|\le 24$?

14. 下图中，$AB=AC=1$，$DE//BC$ 且 $DE:BC=2:3$。P 是线段 DE 上的一点。直线 CP 与直线 BP 的延长线分别交 AB 与 AC 于 G 及 H。求 $\dfrac{120}{BG}+\dfrac{120}{CH}$。

In the figure below, $AB=AC=1$, $DE//BC$ and $DE:BC=2:3$. P is a point on the line segment DE. The extensions of the line CP and the line BP intersect AB and AC respectively at G and H. Find $\dfrac{120}{BG}+\dfrac{120}{CH}$.



15. 设 $n=9\times 99\times 999\times\ldots\times \underbrace{99\cdots9}_{2017\text{个 }9}$，求 n 的最后三位数字。

Let $n=9\times 99\times 999\times\ldots\times \underbrace{99\cdots9}_{2017\text{ digits of }9}$. Find the last three digits of n.

16. 对于任意的实数 k，设 C_k 是圆 $x^2+y^2+kx+ky-2k-10=0$。若对于任何 k，圆 C_k 都经过两个定点 A 及 B，d 是线段 AB 的长，求 d^2。

For any real number k, let C_k be the circle $x^2+y^2+kx+ky-2k-10=0$. If the circles C_k passes through two fixed points A and B for any values of k, and d is the length of the line segment AB, find d^2.

17. 设S是集合$A=\left\{\frac{a}{b}:\ a,b\text{ 是 100 的正因数且 } a \text{ 与 } b \text{ 互质}\right\}$里的各数之和。求$\lfloor S \rfloor$。

Let S be the sum of the elements in the set

$A=\left\{\frac{a}{b}:\ a,b \text{ are positive factors of 100, and } a \text{ and } b \text{ are relatively prime}\right\}$.

Find $\lfloor S \rfloor$.

18. 若$S_n=\frac{1}{1\times2\times4}+\frac{1}{2\times3\times5}+\frac{1}{3\times4\times6}+\ldots+\frac{1}{n\times(n+1)\times(n+3)}$，求$\lim\limits_{n\to\infty}(360S_n)$。

If $S_n=\frac{1}{1\times2\times4}+\frac{1}{2\times3\times5}+\frac{1}{3\times4\times6}+\ldots+\frac{1}{n\times(n+1)\times(n+3)}$, find $\lim\limits_{n\to\infty}(360S_n)$.

19. 求由曲线$\sqrt{|x|}+\sqrt{|y|}=3$所包围区域的面积。

Find the area of the region enclosed by the curve $\sqrt{|x|}+\sqrt{|y|}=3$.

20. 已知平行四边形$OABC$的其中三个顶点为$O(0,0)$，$A(36,84)$及$B(h,k)$，其中h，k为整数，求平行四边形$OABC$的面积的最小可能值。

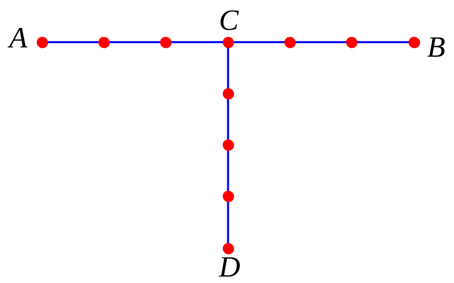
Given that three of the vertices of parallelogram $OABC$ are $O(0,0)$, $A(36,84)$ and $B(h,k)$, where h and k are integers. Find the smallest possible value of the area of parallelogram $OABC$.

第 21 至第 25 题，问答题，每题 6 分。

Question 21 to Question 25, short questions, each question carries 6 marks.

21. 下图中，有 7 个点在线段 AB 上，5 个点在线段 CD 上，其中点 C 是两条线段的交点。由这 11 个点可组成多少个不同的三角形？

In the figure below, there are 7 points on the line segment AB, and there are 5 points on the line segment CD. C is the intersection of the two line segments. How many distinct triangles can be formed from these 11 points?



22. 有多少组正整数 (a, b, c) 满足以下的条件：a 是 b 与 c 的因数且 $a+b+c=45$。

How many triples of positive integers (a, b, c) satisfy the following conditions: a is a factor of b and c, and $a+b+c=45$.

23. 设 a_1， a_2， a_3， …是所有由 0，1，2，5，6，8，9 这七个数字所组成的非负整数按由小到大排列所组成的数列； $a_1=0$， $a_2=1$， $a_3=2$， …， $a_8=10$， $a_9=11$， $a_{10}=12$， …。求 a_{2017} 的最后三位数。

Let a_1, a_2, a_3, …be the sequence of nonnegative integers formed by the 7 digits 0, 1, 2, 5, 6, 8, 9, arranged in ascending order. $a_1=0$, $a_2=1$, $a_3=2$, …, $a_8=10$, $a_9=11$, $a_{10}=12$, …. Find the last three digits of a_{2017}.

24. 设x_1，x_2，x_3，x_4，x_5，x_6是实数且满足

$$\begin{cases} x_1+4x_2+9x_3+16x_4+25x_5+36x_6=30 \\ 4x_1+9x_2+16x_3+25x_4+36x_5+49x_6=100 \\ 9x_1+16x_2+25x_3+36x_4+49x_5+64x_6=250 \end{cases}$$

求$16x_1+25x_2+36x_3+49x_4+64x_5+81x_6$的值。

Let x_1, x_2, x_3, x_4, x_5, x_6 be real numbers satisfying

$$\begin{cases} x_1+4x_2+9x_3+16x_4+25x_5+36x_6=30 \\ 4x_1+9x_2+16x_3+25x_4+36x_5+49x_6=100 \\ 9x_1+16x_2+25x_3+36x_4+49x_5+64x_6=250 \end{cases}$$

Find the value of $16x_1+25x_2+36x_3+49x_4+64x_5+81x_6$.

25. 设$S=|n-1|+2|n-2|+3|n-3|+\ldots+20|n-20|$，其中$n$是正整数。求$S$的最小可能值。

Let $S=|n-1|+2|n-2|+3|n-3|+\ldots+20|n-20|$, where n is a positive integer. Find the minimum possible value of S.

第 26 至第 30 题，问答题，每题 8 分。

Question 26 to Question 30, short questions, each question carries 8 marks.

26. 已知a，b，x，y是实数且

$$\begin{cases} ax+by=2 \\ ax^2+by^2=3 \\ ax^3+by^3=4 \\ ax^4+by^4=7 \end{cases}$$

求ax^5+by^5的值。

Given that a, b, x, y are real numbers and

$$\begin{cases} ax+by=2 \\ ax^2+by^2=3 \\ ax^3+by^3=4 \\ ax^4+by^4=7 \end{cases}$$

Find the value of ax^5+by^5.

27. 设$n=1+11+111+\ldots+\underbrace{111\ldots111}_{2016\text{个}1}$，则整数$n$里有几位数是1？

Let $n=1+11+111+\ldots+\underbrace{111\ldots111}_{2016\text{ digits of }1}$. Then how many digits in n are equal to 1?

28. 若x，y，z是正数且$x+y+z=6$，求x^3y^2z的最大可能值。

If x, y, z are positive numbers such that $x+y+z=6$, find the maximum possible value of x^3y^2z.

29. 若n是正整数，令$f(n)$为最靠近$\sqrt{n}$的正整数。例如：$f(1)=1$，$f(2)=1$，$f(3)=2$，$f(4)=2$等。求$\sum_{n=1}^{\infty}\frac{\left(\frac{4}{3}\right)^{f(n)}+1+\left(\frac{4}{3}\right)^{-f(n)}}{\left(\frac{4}{3}\right)^{n}}$。

If n is a positive integer, let $f(n)$ be the positive integer closest to $\sqrt{n}$. For example, $f(1)=1$, $f(2)=1$, $f(3)=2$, $f(4)=2$. Find $\sum_{n=1}^{\infty}\frac{\left(\frac{4}{3}\right)^{f(n)}+1+\left(\frac{4}{3}\right)^{-f(n)}}{\left(\frac{4}{3}\right)^{n}}$.

30. 考虑以下的两个条件：

(a) n是小于1000的正整数；

(b) n^n+1是66的倍数。

满足这两个条件的正整数n有几个？

Consider the following two conditions:

(a) n is a positive integer that is less than 1000.

(b) n^n+1 is a multiple of 66.

How many positive integers n satisfy both these conditions?