厦门大学马来西亚分校
陈景润杯中学数学比赛

2018 年第 2 届陈景润杯中学数学比赛

~ 高阶组 ~

日期：2018 年 4 月 21 日

Date: 21st April 2018

时间：上午 10 时至中午 12 时

Time: 10:00 a.m. to 12:00 p.m.

考生须知
Instructions and Information

1. 本试卷共有 30 题。
   This paper contains 30 questions.
   - 第 1 题至第 10 题，选择题，每题 4 分。
     Question 1 to Question 10, multiple choice questions, each question carries 4 marks.
   - 第 11 题至第 30 题，问答题，每题的答案是一个介于 0 至 1000 之间的整数。
     Question 11 to Question 30, short questions. For each question, the answer is an integer between 0 and 1000.
     - 第 11 题至第 20 题每题 5 分。
       Question 11 to Question 20, each question carries 5 marks.
     - 第 21 题至第 25 题每题 6 分。
       Question 21 to Question 25, each question carries 6 marks.
     - 第 26 题至第 30 题每题 8 分。
       Question 26 to Question 30, each question carries 8 marks.
2. 请在答案纸内适当的空格中用 2B 铅笔清楚的写出每题的答案。对于选择题，只需填写 A，B，C，D 或 E 作为答案。每题只能填入一个答案，否则以答错论。
   Please use 2B lead pencils to write your answer to each question in the appropriate space provided on the answer sheet. For a multiple choice question, you only need to write A, B, C, D or E as answer. Only one answer is accepted for each question, otherwise no credits would be given.
3. 所有的图形并没有按照比例作图，只作为辅助之用。
   All the diagrams are not drawn to scales. They are intended only as aids.
4. 不许使用计算器，数学工具，手机或其他计算器。
   No calculators, maths stencils, mobile phones or other calculating aids are permitted.
5. 在答案纸上清楚写上姓名，考生编号，学校名称及在学年级。
   Write your name, candidate number, school name and year of study clearly on the answer sheet.
6. 在监考老师宣布比赛开始之后，才可以翻开此考卷开始作答。
   You can only open this question booklet to start answering questions after the invigilator announces the beginning of the competition.

~~ 说明 ~~

~~ Notes ~~

在这份试卷中，$\lfloor x \rfloor$ 表示小于或等于 x 的最大整数。

例如：$\lfloor 2 \rfloor = 2$，$\lfloor -2 \rfloor = -2$，$\lfloor 2.6 \rfloor = 2$，$\lfloor -2.6 \rfloor = -3$。

In this paper, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

For example, $\lfloor 2 \rfloor = 2$, $\lfloor -2 \rfloor = -2$, $\lfloor 2.6 \rfloor = 2$, $\lfloor -2.6 \rfloor = -3$.

第 1 至第 10 题，选择题，每题 4 分。

Question 1 to Question 10, multiple choice questions, each question carries 4 marks.

1. 若现在是星期六下午 3 时正，2018 小时后是星期几?
   If it is Saturday and 3:00 p.m. now, what day is it 2018 hours later?

   A. 星期天 Sunday　B. 星期二 Tuesday　C. 星期四 Thursday　D. 星期五 Friday　E. 星期六 Saturday

2. 求 $\dfrac{2-\sqrt{5}}{3+\sqrt{5}}+\dfrac{2+\sqrt{5}}{3-\sqrt{5}}$。
   Find $\dfrac{2-\sqrt{5}}{3+\sqrt{5}}+\dfrac{2+\sqrt{5}}{3-\sqrt{5}}$.

   A. $\dfrac{5\sqrt{5}}{2}$　B. $5\sqrt{5}$　C. $\dfrac{11}{2}$　D. 11　E. 22

3. 介于 990 与 1000 的整数中，有多少个是质数?
   Among the integers between 990 and 1000, how many of them are prime numbers?

   A. 0　B. 1　C. 2　D. 3　E. 4

4. 循环小数 $2.2\dot{4}\dot{5} = 2.245454545\cdots$ 等于以下哪一个分数?
   The repeating decimal $2.2\dot{4}\dot{5} = 2.245454545\cdots$ is equal to which of the following fractions?

   A. $\dfrac{2243}{999}$　B. $\dfrac{2243}{990}$　C. $\dfrac{2245}{990}$　D. $\dfrac{2223}{990}$　E. $\dfrac{2223}{999}$

5. 一班里有 47 位学生。一次考试后，老师将他们的分数按由小到大的顺序排列为 a_1，a_2，a_3，…，a_{47}。结果老师发现一个规律: 对于所有的 $1 \le n \le 47$，

$$a_1 + a_2 + a_3 + \ldots + a_n = n(n+3)$$

最高分是几分?

There are 47 students in a class. After an examination, the teacher arranged the marks in ascending order as a_1, a_2, a_3, …, a_{47}. Then, the teacher discovered a pattern: for all $1 \le n \le 47$,

$$a_1 + a_2 + a_3 + \ldots + a_n = n(n+3)$$

What is the highest mark scored?

A. 96 B. 97 C. 98 D. 99 E. 100

6. 一个箱子里有 5 粒红球，6 粒黄球及 7 粒白球。若晓兰任意从箱子里拿出若干粒球，则她至少需拿出多少粒球，才能保证至少有两粒球是同样颜色的?

In a box, there are 5 red, 6 yellow and 7 white balls. If Xiao Lan randomly takes out a number of balls from the box, at least how many balls she should take in order to guarantee herself two balls of the same colour?

A. 4 B. 5 C. 6 D. 7 E. 8

7. 已知 $4 \le x \le 6$，化简 $\sqrt{4x^2 - 28x + 49} + \sqrt{4x^2 - 52x + 169}$。

Given that $4 \le x \le 6$, simplify $\sqrt{4x^2 - 28x + 49} + \sqrt{4x^2 - 52x + 169}$.

A. $4x - 20$ B. 20 C. $4x + 20$ D. $8x - 20$ E. 6

8. 有多少对正整数 (a, b) 使得 a 与 b 的最大公因数是 $2 \times 3 \times 5 \times 7$，$a$ 与 b 的最小公倍数是 $2^2 \times 3^2 \times 5^2 \times 7$?

How many pairs of positive integers (a, b) are such that the greatest common divisor of a and b is $2 \times 3 \times 5 \times 7$, and the least common multiple of a and b is $2^2 \times 3^2 \times 5^2 \times 7$?

A. 1 B. 2 C. 4 D. 8 E. 16

9. 有 3 位学生，每人将一份礼物交给老师后，老师又任意的将这 3 份礼物发给这 3 位学生，每人一份。求没有学生拿回自己的礼物的概率。
3 students, each gives a gift to their teacher. The teacher then randomly distributes the three gifts to these three students, each student one. Find the probability that no student gets back his own gift.

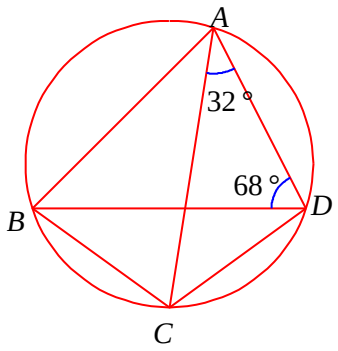
A. $\frac{1}{6}$ B. $\frac{1}{3}$ C. $\frac{1}{2}$ D. $\frac{2}{3}$ E. $\frac{5}{6}$

10. 今年学校来了 6 位高中一的插班生，教务主任要将他们安插在 3 班高中一的班级中，每班两位。有多少种方法?
This year there are 6 new students joining the senior one classes. The head of academics would like to put them into 3 senior one classes, 2 students each. How many ways are there?

A. 15 B. 21 C. 90 D. 120 E. 720

第 11 至第 20 题，问答题，每题 5 分。
Question 11 to Question 20, short questions, each question carries 5 marks.

11. 下图中，$ABCI$ 是圆内接四边形，$CB=CD$，$\angle CAD=32°$，$\angle ADB=68°$。若 $\angle ABD=x°$，求 x。
In the figure below, $ABCD$ is a cyclic quadrilateral, $CB=CD$, $\angle CAD=32°$, $\angle ADB=68°$. If $\angle ABD=x°$, find x.

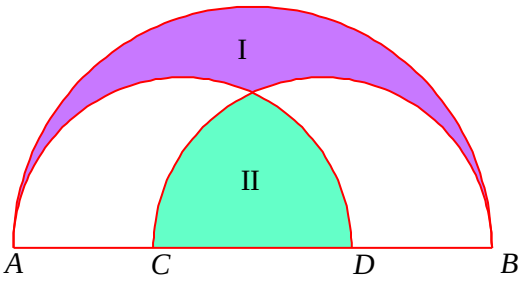


12. 17 位学生，编号 1 至 17，按编号顺序围成一个圆圈。他们开始喊号码，编号 1 的学生喊 1，编号 2 的喊 2，依此类推。编号 17 的学生喊 17 后，编号 1 的学生接下去喊 18，编号 2 的喊 19，如此不断继续下去。喊 2018 的学生编号是多少？

17 students, numbered 1 to 17, formed a circle according to their numerical order. They began to count out loud. Student no. 1 called out 1, student no. 2 called out 2, and so on. After student no. 17 called out 17, student no. 1 continued to call out 18, student no. 2 called out 19, and so on. This process continues. What is the number of the student who called out 2018?

13. 下图所示为分别以 AB，AD 及 BC 为直径的半圆，其中 $AB=d_1$，$AD=BC=d_2$。若阴影部分 I 的面积与阴影部分 II 的面积相等，求 $\dfrac{12d_1^2}{d_2^2}$。

The figure below shows three semicircles with diameters AB, AD and BC respectively, with $AB=d_1$, $AD=BC=d_2$. If the area of the shaded region I is the same as the area of the shaded region II, find $\dfrac{12d_1^2}{d_2^2}$.



14. 求 $\left\lfloor\dfrac{3^2}{1}\right\rfloor+\left\lfloor\dfrac{4^2}{2}\right\rfloor+\left\lfloor\dfrac{5^2}{3}\right\rfloor+\left\lfloor\dfrac{6^2}{4}\right\rfloor+\ldots+\left\lfloor\dfrac{42^2}{40}\right\rfloor$。

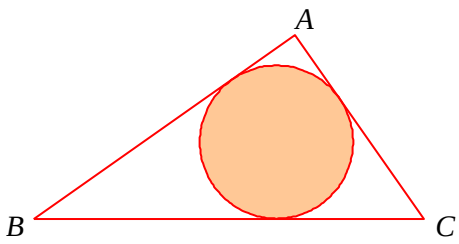
Find $\left\lfloor\dfrac{3^2}{1}\right\rfloor+\left\lfloor\dfrac{4^2}{2}\right\rfloor+\left\lfloor\dfrac{5^2}{3}\right\rfloor+\left\lfloor\dfrac{6^2}{4}\right\rfloor+\ldots+\left\lfloor\dfrac{42^2}{40}\right\rfloor$.

15. 已知 $2x^2-7x+4=0$，求 $41x-4x^3$ 的值。

Given that $2x^2-7x+4=0$. Find the value of $41x-4x^3$.

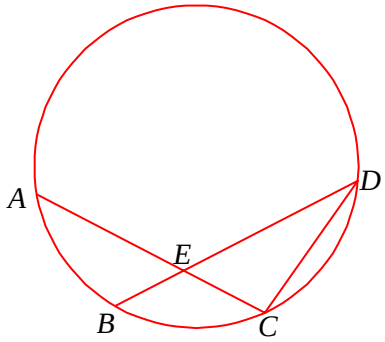
16. 下图所示为直角 ΔABC 及其内切圆。已知 $\angle A$ 是直角，$AB=15$，$AC=8$，阴影部分 (内切圆) 的面积为 $S\pi$，求 $42S$ 的值。

The figure below shows a right-angled triangle ΔABC and its inscribed circle. Given that $\angle A$ is a right angle, $AB=15$, $AC=8$, and the area of the shaded region (the inscribed circle) is $S\pi$, find the value of $42S$.



17. 下图中，$\overset{\frown}{AB}=\overset{\frown}{BC}=\overset{\frown}{CD}$。若 $\angle AED=124°$，$\angle ACD=x°$，求 x。

In the figure below, $\overset{\frown}{AB}=\overset{\frown}{BC}=\overset{\frown}{CD}$. If $\angle AED=124°$, $\angle ACD=x°$, find x.



18. 若 x，y，z 是正数使得

$$\frac{x}{\sqrt{x^2+y^2+z^2}}=\frac{1}{2},\quad \frac{y}{\sqrt{x^2+y^2+z^2}}=\frac{2}{3},\quad \frac{z}{\sqrt{x^2+y^2+z^2}}=\frac{\sqrt{a}}{6}$$

求 a 的值。

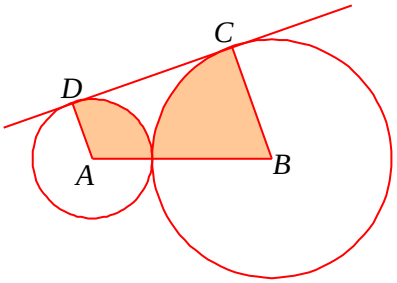
If x, y, z are positive numbers such that

$$\frac{x}{\sqrt{x^2+y^2+z^2}}=\frac{1}{2},\quad \frac{y}{\sqrt{x^2+y^2+z^2}}=\frac{2}{3},\quad \frac{z}{\sqrt{x^2+y^2+z^2}}=\frac{\sqrt{a}}{6}$$

Find the value of a.

19. 下图中，两圆相切，它们的圆心分别为点 A 及点 B。直线 CD 是两圆的公切线，点 C 及点 D 是切点。若两圆的半径分别为 12 及 36，阴影部分的面积为 $a\pi$，求 $\lfloor a \rfloor$。

In the figure below, the two circles are tangent to each other. The centers of the circles are A and B respectively. The line CD is a common tangent to the two circles. C and D are points of tangency. If the radii of the two circles are 12 and 36 respectively, and the area of the shaded region is $a\pi$, find $\lfloor a \rfloor$.



20. 在一次周会，老师要学生们排队。参与的学生人数少于 1000 人。无论是 15 人排一行，或 18 人排一行，或 20 人排一行，都会多出一位学生。问参与周会的学生最多有多少人?

In a weekly assembly, the teacher asks the students to line up in rows. The number of students at the assembly is less than 1000. If 15 students line up in a row, or 18 students line up in a row, or 20 students line up in a row, there will always be one student extra. At most how many students are there at the assembly?

第 21 至第 25 题，问答题，每题 6 分。

Question 21 to Question 25, short questions, each question carries 6 marks.

21. 一函数 f 满足 $f(x)+f(3x+1)=x^2$。若 $f(3)+f(31)=63$，求 $f(10)$的值。

A function f satisfies $f(x)+f(3x+1)=x^2$. If $f(3)+f(31)=63$, find the value of $f(10)$.

22. 若 x， y 是实数且满足 $2x^2+5y^2=7x$，求 $18x+10y^2$ 的最大可能值。

If x and y are real numbers such that $2x^2+5y^2=7x$, find the largest possible value of $18x+10y^2$.

23. 已知 $a+b+c=6$，求 $2\times3^a+16\times3^{b-1}+16\times3^{c-2}$ 的最小可能值。

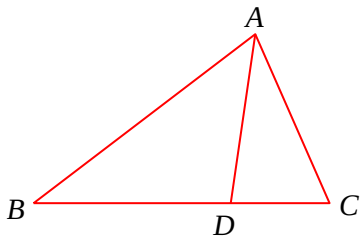
Given that $a+b+c=6$, find the smallest possible value of $2\times3^a+16\times3^{b-1}+16\times3^{c-2}$.

24. 已知多项式 $f(x)$，它除以 (x^2+1) 得余式 $(2x+3)$，除以 (x^2-1) 得余式 $(-4x+13)$，除以 (x^4-1) 得余式 $r(x)$，求 $r(-3)$ 的值。

Given the polynomial $f(x)$, it leaves a remainder of $(2x+3)$ when divided by (x^2+1), it leaves a remainder of $(-4x+13)$ when divided by (x^2-1), and it leaves a remainder of $r(x)$ when divided by (x^4-1). Find the value of $r(-3)$.

25. 下图中，$BD:CD=2:1$，$AB=20$，$AC=13$，$BC=18$，$AD=x$。求 x^2。

In the figure below, $BD:CD=2:1$, $AB=20$, $AC=13$, $BC=18$, $AD=x$. Find x^2.



第 26 至第 30 题，问答题，每题 8 分。

Question 26 to Question 30, short questions, each question carries 8 marks.

26. 若 $(x+1)(x+2)(x+3)\cdots(x+20)$ 的展开式中 x^{18} 的系数是 a，求 a 的最后三位数。

If the coefficient of x^{18} in the expansion of $(x+1)(x+2)(x+3)\cdots(x+20)$ is a, find the last three digits of a.

27. 已知一数列 a_1，a_2，a_3，…的定义为

$a_1=1$，$a_2=4$，且对于所有 $n\geq 3$，$a_n=2a_{n-1}-a_{n-2}+n$

设 $x=\dfrac{a_{96}-a_{94}}{12}$。求 x 的值。

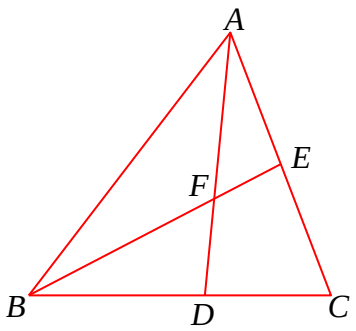
Given that a sequence $a_1, a_2, a_3, \ldots$ is defined as

$a_1=1$, $a_2=4$, and $a_n=2a_{n-1}-a_{n-2}+n$ for all $n\geq 3$

Let $x=\dfrac{a_{96}-a_{94}}{12}$. Find the value of x.

28. 下图中，D，E 分别是直线 BC 及 AC 上的点，$BD:DC=9:7$，$AE:EC=4:5$。直线 AD 与 BE 相交于点 F。若 $\dfrac{AF}{FD}=x$，求 $360x$ 的值。

In the figure shown below, D, E are respectively points on the line BC and AC such that $BD:DC=9:7$ and $AE:EC=4:5$. The lines AD and BE intersect at the point F. If $\dfrac{AF}{FD}=x$, find the value of $360x$.



29. 求 $\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\sum_{n=0}^{\infty}\frac{300}{2^n\times k+5}$。

Find $\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\sum_{n=0}^{\infty}\frac{300}{2^n\times k+5}$.

30. 有多少组正整数 (a,b,c,d) 满足 $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=1$?

How many quadruples of positive integers (a,b,c,d) satisfy $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=1$?