



廈門大學馬來西亞分校
陳景潤杯中學數學比賽



CHEN JINGRUN'S CUP SECONDARY SCHOOL
MATHEMATICS COMPETITION 2019

**** 高阶组 ****
SENIOR CATEGORY

日期: 2019年4月9日

时间: 上午10时至中午12时

Date: 9th April 2019

Time: 10:00 a.m. to 12:00 p.m.

考生须知

Instructions and Information

1. 本试卷共有30题。
This paper contains 30 questions.
 - 第1题至第10题, 选择题, 每题4分。
Question 1 to Question 10, multiple choice questions, each question carries 4 marks.
 - 第11题至第30题, 问答题, 每题的答案是一个介于0至1000之间的整数。
Question 11 to Question 30, short questions. For each question, the answer is an integer between 0 and 1000.
 - 第11题至第20题每题5分。
Question 11 to Question 20, each question carries 5 marks.
 - 第21题至第25题每题6分。
Question 21 to Question 25, each question carries 6 marks.
 - 第26题至第30题每题8分。
Question 26 to Question 30, each question carries 8 marks.
2. 请在答案纸内适当的空格中用2B铅笔清楚的写出每题的答案。对于选择题, 必须填写A, B, C, D或E作为答案。每题只能填入一个答案, 否则以答错论。
Please use 2B pencils to write your answers in the appropriate boxes provided on the answer sheet. For each multiple choice question, please write A, B, C, D or E as answer. If more than one answer is found for a question, no credits would be given for that question.
3. 所有的图形并没有按照比例作图, 只作为辅助之用。
All the diagrams are not drawn to scale. They are intended as aids only.
4. 不许使用计算器, 数学工具, 手机或其他计算器。
No calculators, maths stencils, mobile phones or other calculating aids are permitted.
5. 在答案纸上清楚写上姓名, 考生编号, 学校名称及就读年级。
Write your name, candidate number, name of school and year of study clearly on the answer sheet.
6. 在监考老师宣布比赛开始之后, 才可以翻开此考卷开始作答。
Do not open this question booklet until you are told to do so.

~~ 说明 ~~

~~ Notes ~~

在这份试卷中, $\lfloor x \rfloor$ 表示小于或等于 x 的最大整数。

例如: $\lfloor 2 \rfloor = 2$, $\lfloor -2 \rfloor = -2$, $\lfloor 2.6 \rfloor = 2$, $\lfloor -2.6 \rfloor = -3$ 。

In this paper, $\lfloor x \rfloor$ denotes the largest integer less than or equal to x .

For example, $\lfloor 2 \rfloor = 2$, $\lfloor -2 \rfloor = -2$, $\lfloor 2.6 \rfloor = 2$, $\lfloor -2.6 \rfloor = -3$.

第 1 至第 10 题, 选择题, 每题 4 分。

Question 1 to Question 10, multiple choice questions, each question carries 4 marks.

1. 已知多项式 $p(x)$ 除以 $2x^2 - 3x - 2$ 的余式是 $4x - 7$, 求 $p(x)$ 除以 $2x + 1$ 的余数。

Given that the polynomial $p(x)$ leaves a remainder of $4x - 7$ when it is divided by $2x^2 - 3x - 2$, find the remainder when $p(x)$ is divided by $2x + 1$.

A. 9 B. 5 C. -5 D. -8 E. -9

$$2x^2 - 3x - 2 = (2x + 1)(x - 2)$$

$$p(x) = (2x^2 - 3x - 2)q(x) + 4x - 7$$

$$p\left(-\frac{1}{2}\right) = 0 + 4\left(-\frac{1}{2}\right) - 7 = -9$$

ANSWER: 【E】

2. 一个箱子中有 9 粒白球, 7 粒红球。任意从箱子中取出一球给林慧, 再取出一球给李明。求李明拿到的球是红球的概率。

There are 9 white balls and 7 red balls in a box. A ball is randomly taken out from the box and passed to LinHui, then a second ball is taken out and passed to LiMing. Find the probability that the ball obtained by LiMing is a red ball.

- A. $\frac{7}{16}$ B. $\frac{9}{16}$ C. $\frac{2}{5}$ D. $\frac{8}{15}$ E. $\frac{7}{15}$

Method 1:

The sample space has 16×15 elements. Among them, there are 7×15 for which the second ball is red.

Hence, the probability that the second ball is red is $\frac{7 \times 15}{16 \times 15} = \frac{7}{16}$.

Method 2:

LinHui obtains a white ball with probability $\frac{9}{16}$. In this case, LiMing obtains a red ball with probability $\frac{7}{15}$.

LinHui obtains a red ball with probability $\frac{7}{16}$. In this case, LiMing obtains a red ball with probability $\frac{6}{15}$.

Hence, the probability that LiMing gets a red ball is $\frac{9}{16} \times \frac{7}{15} + \frac{7}{16} \times \frac{6}{15} = \frac{7}{16}$.

ANSWER: 【A】

3. 波波，东东，卡卡三人一起去超市。波波买了一粒苹果，两粒橙及三粒芒果，花了 RM17.60。东东买了两粒苹果，三粒橙，一粒芒果，花了 RM12.00。卡卡买了三粒苹果，一粒橙，两粒芒果，花了 RM14.20。问一粒苹果的单价是多少？

Bobo, Dongdong, Kaka went to a grocery store together. Bobo spent RM17.60 to buy an apple, two oranges and three mangos. Dongdong spent RM12.00 to buy two apples, three oranges and one mango. Kaka spent RM14.20 to buy three apples, one orange and two mangos. What is the unit price of an apple?

- A. RM1.30 B. RM1.40 C. RM1.50 D. RM1.60 E. RM1.70

Let RM x be the unit price of an apple, RM y be the unit price of an orange, and RM z be the unit price of a mango. Then

$$\begin{cases} x+2y+3z=17.6 \\ 2x+3y+z=12.0 \\ 3x+y+2z=14.2 \end{cases} \Rightarrow \begin{cases} 5x+7y=18.4 \\ x+5y=9.8 \end{cases} \Rightarrow 18y=30.6$$

$$\therefore y=1.7, \quad x=1.3$$

ANSWER: 【A】

4. 当一粒球的半径增加了10%，求它的体积增加的百分比。

When the radius of a ball is increased by 10%, find the percentage increase in its volume.

- A. 20% B. 21% C. 30% D. 33% E. 33.1%

$$r_2 = 1.1 r_1$$

$$\begin{aligned} V_2 &= \frac{4}{3} \pi r_2^3 \\ &= \frac{4}{3} \pi (1.1 r_1)^3 \\ &= 1.1^3 V_1 \\ &= 1.331 V_1 \end{aligned}$$

Namely, the volume increased by 33.1%.

ANSWER: 【E】

5. 求 $\sqrt[3]{4035^3 - 2017^3 - 3 \times 4035 \times 2017 \times 2018}$ 。

Find $\sqrt[3]{4035^3 - 2017^3 - 3 \times 4035 \times 2017 \times 2018}$.

- A. 2017 B. 2018 C. 2019 D. 2020 E. 2021

Let $x = 4035$, $y = 2017$.

$$\begin{aligned}\sqrt[3]{4035^3 - 2017^3 - 3 \times 4035 \times 2017 \times 2018} &= \sqrt[3]{x^3 - y^3 - 3xy(x - y)} \\ &= \sqrt[3]{x^3 - 3x^2y + 3xy^2 - y^3} \\ &= \sqrt[3]{(x - y)^3} \\ &= x - y \\ &= 2018\end{aligned}$$

ANSWER: 【B】

6. 已知组合数 $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ 是由 n 件不同的物品选出 k 件的方法数。化简

$$\binom{37}{1} - \binom{37}{2} + \binom{37}{3} - \binom{37}{4} + \cdots + \binom{37}{17} - \binom{37}{18}$$

Given that the combination number $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the number of ways to choose k

objects from n different objects. Simplify

$$\binom{37}{1} - \binom{37}{2} + \binom{37}{3} - \binom{37}{4} + \cdots + \binom{37}{17} - \binom{37}{18}$$

A. $\binom{36}{18}$

B. $-\binom{36}{18}$

C. $1 + \binom{36}{18}$

D. $1 - \binom{36}{18}$

E. $\binom{36}{18} - 1$

Notice that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

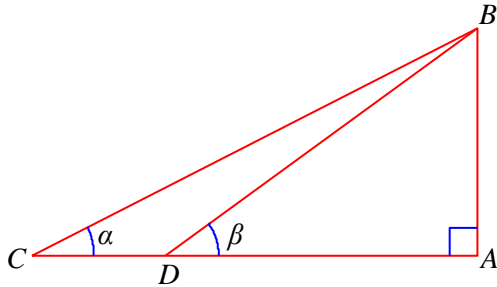
Hence,

$$\begin{aligned} & \binom{37}{1} - \binom{37}{2} + \binom{37}{3} - \binom{37}{4} + \cdots + \binom{37}{17} - \binom{37}{18} \\ &= \binom{36}{0} + \binom{36}{1} - \binom{36}{1} - \binom{36}{2} + \binom{36}{2} + \binom{36}{3} - \binom{36}{3} - \binom{36}{4} + \cdots + \binom{36}{16} + \binom{36}{17} - \binom{36}{17} - \binom{36}{18} \\ &= 1 - \binom{36}{18} \end{aligned}$$

ANSWER: 【D】

7. 下图中, $\triangle ABC$ 是直角三角形, $\angle A$ 是直角, D 是 AC 上的一点。 $\angle BCA = \alpha$, $\angle BDA = \beta$ 。若 $CD = d$, $AB = h$, 则

In the figure below, $\triangle ABC$ is a right-angled triangle with $\angle A$ being the right angle. D is a point on AC , and $\angle BCA = \alpha$, $\angle BDA = \beta$. If $CD = d$, $AB = h$, then



- A. $d = h(\sin \beta - \sin \alpha)$
 B. $d = h(\tan \beta - \tan \alpha)$
 C. $d = h(\cot \alpha - \cot \beta)$
 D. $d = h(\cos \alpha - \cos \beta)$
 E. $d = h(\sec \beta - \sec \alpha)$

$$\frac{AB}{AC} = \tan \alpha \quad \Rightarrow \quad AC = h \cot \alpha$$

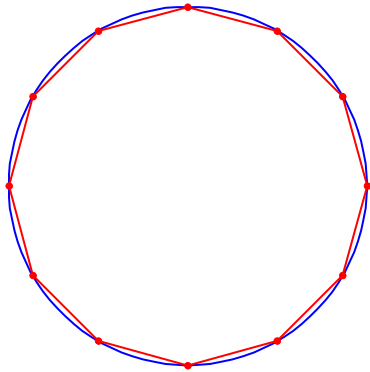
$$\frac{AB}{AD} = \tan \beta \quad \Rightarrow \quad AD = h \cot \beta$$

$$d = h(\cot \alpha - \cot \beta)$$

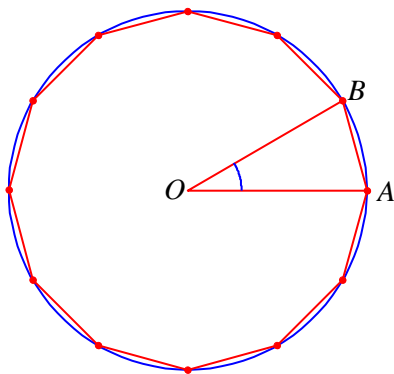
ANSWER: 【C】

8. 下图所示为一圆与一内接于圆中的正十二边形。若圆的半径为 $\sqrt{6} + \sqrt{2}$ ，求正十二边形的周长。

The figure below shows a circle and an inscribed regular 12-gon. If the radius of the circle is $\sqrt{6} + \sqrt{2}$, find the perimeter of the regular 12-gon.



- A. 12 B. 18 C. 24 D. 36 E. 48



$$\angle AOB = \frac{2\pi}{12} = \frac{\pi}{6}$$

$$r = \sqrt{6} + \sqrt{2}$$

$$AB^2 = r^2 + r^2 - 2r^2 \cos \frac{\pi}{6}$$

$$= \frac{r^2}{2} (4 - 2\sqrt{3})$$

$$AB = \frac{r}{\sqrt{2}} (\sqrt{3} - 1)$$

$$= (\sqrt{3} + 1)(\sqrt{3} - 1)$$

$$= 2$$

Therefore, the perimeter of the regular 12-gon is 24.

ANSWER: 【C】

9. 已知 n 是整数且 $n(\sqrt{399} - \sqrt{397}) < 1$ 。求 n 的最大可能值。

Given that n is an integer such that $n(\sqrt{399} - \sqrt{397}) < 1$. Find the largest possible value of n .

- A. 18 B. 19 C. 38 D. 39 E. 40

$$\begin{aligned} n &< \frac{1}{\sqrt{399} - \sqrt{397}} \\ &= \frac{\sqrt{399} + \sqrt{397}}{2} \end{aligned}$$

Since $19 < \sqrt{397} < 20$, $19 < \sqrt{399} < 20$, we find that $19 < \frac{\sqrt{399} + \sqrt{397}}{2} < 20$.

Hence, the largest possible value of n is 19.

ANSWER: 【B】

10. 将一个边长为 13 cm 的立方体的表面涂上红色，再将它切割成边长为 1 cm 的小立方体。在这些小立方体中，有多少个恰好有两面被涂上红色？

After the faces of a cube with side length 13 cm are colored red, the cube is dissected into small cubes, each with side length 1 cm. Among the small cubes, how many of them have exactly two red faces?

- A. 169 B. 156 C. 144 D. 132 E. 121

The small cubes with two red faces are those originally lying at the interior of each edge of the large cube. Since a cube has 12 edges, each edge gives rise to 11 small cubes with two red faces, there are altogether $12 \times 11 = 132$ small cubes with two red faces.

ANSWER: 【D】

第 11 至第 20 题，问答题，每题 5 分。

Question 11 to Question 20, short questions, each question carries 5 marks.

11. 已知 131 个数 x_1, x_2, \dots, x_{131} 的平均数是 131。对于 $1 \leq k \leq 131$, $y_k = x_k + k$, 求 y_1, y_2, \dots, y_{131} 这 131 个数的平均数。

Given that the average of the 131 numbers x_1, x_2, \dots, x_{131} is 131. If $y_k = x_k + k$ for $1 \leq k \leq 131$, find the average of the 131 numbers y_1, y_2, \dots, y_{131} .

The average of the 131 numbers y_1, y_2, \dots, y_{131} is

$$\begin{aligned}\frac{y_1 + y_2 + \dots + y_{131}}{131} &= \frac{x_1 + x_2 + \dots + x_{131}}{131} + \frac{1 + 2 + \dots + 131}{131} \\ &= 131 + \frac{1 + 131}{2} \\ &= 197\end{aligned}$$

ANSWER: 【197】

12. 求满足不等式 $\log_{\frac{1}{2}}(5n - 43) + 8 \geq 0$ 的最大整数 n 。

Find the largest integer n that satisfies the inequality $\log_{\frac{1}{2}}(5n - 43) + 8 \geq 0$.

$$\begin{aligned}\log_{\frac{1}{2}}(5n - 43) &\geq -8 \\ 5n - 43 &\leq \left(\frac{1}{2}\right)^{-8} \\ 5n - 43 &\leq 256 \\ 5n &\leq 299 \\ n &\leq 59\end{aligned}$$

ANSWER: 【59】

13. 有一群人参加会议。第一阶段结束后，66 位女士离开会场，使得会场中所剩下的男、女之比例为 5:3。第二阶段结束后，又有 132 位男士离开会场，使得会场中所剩下的男、女之比例为 3:4。问会议刚开始时有多少位女士在会场？

A group of people participated in a conference. After the first session, 66 of the female participants left, yielding a ratio of male to female participants in the conference room to be 5:3. After the second session, another 132 male participants left, yielding a ratio of male to female participants in the conference room to be 3:4. How many female participants were there at the beginning of the conference?

Let x and y be respectively the numbers of male and female participants at the beginning of the conference. Then

$$\frac{x}{y-66} = \frac{5}{3}$$

$$\frac{x-132}{y-66} = \frac{3}{4}$$

$$\frac{5}{3}(y-66) = \frac{3}{4}(y-66) + 132$$

$$\frac{11}{12}(y-66) = 132$$

$$y = 210$$

ANSWER: 【210】

14. 已知圆 $x^2 + y^2 + 4x - 18y - 60 = 0$ 与直线 $x - 2y = 5$ 相交于点 A 及点 B ，点 C 是圆的圆心。求 $\triangle ABC$ 的面积。

Given that the circle $x^2 + y^2 + 4x - 18y - 60 = 0$ intersects the line $x - 2y = 5$ at points A and B , and C is the center of the circle. Find the area of $\triangle ABC$.

$$x^2 + y^2 + 4x - 18y - 60 = 0$$

$$(x+2)^2 + (y-9)^2 = 145$$

C is the point $(-2, 9)$. The radius of the circle is $\sqrt{145}$.

$$\text{The distance from } C \text{ to } AB \text{ is } h = \left| \frac{-2-18-5}{\sqrt{1+4}} \right| = 5\sqrt{5}.$$

$$\text{The length of } AB \text{ is } 2\sqrt{145-125} = 4\sqrt{5}.$$

$$\text{The area of } \triangle ABC \text{ is } \frac{1}{2} \times 4\sqrt{5} \times 5\sqrt{5} = 50.$$

ANSWER: 【50】

15. 已知 $N = 2^2 \times 3 \times 5^2 \times 6^3 \times 7^4$ 。求 N 的正因子的个数。

Given that $N = 2^2 \times 3 \times 5^2 \times 6^3 \times 7^4$. Find the number of positive factors of N .

The prime factorization of N is

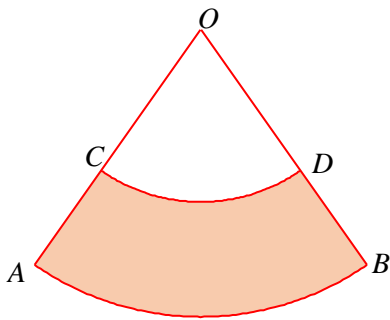
$$N = 2^5 \times 3^4 \times 5^2 \times 7^4$$

Hence, N has $6 \times 5 \times 3 \times 5 = 450$ positive factors.

ANSWER: 【450】

16. 下图中, OAB 及 OCD 是扇形。若 $\widehat{AB} = 20\pi$, $\widehat{CD} = 12\pi$, $AC = 14$, 阴影部分的面积为 $S\pi$, 求 S 的值。

In the figure below, OAB and OCD are circular sectors. If $\widehat{AB} = 20\pi$, $\widehat{CD} = 12\pi$, $AC = 14$, and the area of the shaded region is $S\pi$, find the value of S .



Let $OC = r_1$, $OA = r_2$, $\angle AOB = \theta$. Then

$$AC = r_2 - r_1 = 14$$

$$\widehat{AB} = 20\pi = r_2\theta$$

$$\widehat{CD} = 12\pi = r_1\theta$$

$$\begin{aligned} S\pi &= \frac{1}{2}r_2^2\theta - \frac{1}{2}r_1^2\theta \\ &= \frac{1}{2}(r_2 - r_1)(r_2 + r_1)\theta \\ &= \frac{1}{2} \times 14 \times (20\pi + 12\pi) \\ &= 224\pi \\ S &= 224 \end{aligned}$$

ANSWER: 【224】

17. 已知 $A = \{1, 2, 3, 4\}$, $B = \{5, 6, 7, 8, 9, 10, 11\}$ 。有多少个由 A 到 B 的一对一函数?

Given that $A = \{1, 2, 3, 4\}$, $B = \{5, 6, 7, 8, 9, 10, 11\}$. How many one-to-one functions are there from A to B ?

If $f: A \rightarrow B$ is a one-to-one function, there are 7 choices for $f(1)$. After $f(1)$ is chosen, there are 6 choices for $f(2)$. After that, there are 5 choices for $f(3)$. And after that, there are 4 choices for $f(4)$.

Hence, there are $7 \times 6 \times 5 \times 4 = 840$ one-to-one functions from A to B .

ANSWER: 【840】

18. 求函数 $f(x) = 303 - \frac{200x^2}{x^4 + 16}$ 的最小值。

Find the minimum value of the function $f(x) = 303 - \frac{200x^2}{x^4 + 16}$.

By Arithmetic Mean-Geometric Mean Inequality,

$$x^4 + 16 \geq 2\sqrt{x^4 \times 16} = 8x^2$$

Hence,

$$\frac{200x^2}{x^4 + 16} \leq \frac{200}{8} = 25$$

$$f(x) = 303 - \frac{200x^2}{x^4 + 16} \geq 303 - 25 = 278$$

The minimum value 278 is achieved when $x = 2$.

ANSWER: 【278】

19. 已知 x 是一整数且 $\left\lfloor \frac{x-128}{7} \right\rfloor = -3$, 求 x 的最大可能值。

Given that x is an integer and $\left\lfloor \frac{x-128}{7} \right\rfloor = -3$, find the largest possible value of x .

$$\left\lfloor \frac{x-128}{7} \right\rfloor = -3$$

$$-3 \leq \frac{x-128}{7} < -2$$

$$-21 \leq x-128 < -14$$

$$107 \leq x < 114$$

The largest possible value of x is 113.

ANSWER: 【113】

20. 若 α , β , γ 是方程式 $3x^3 + 12x^2 - 77x + 11 = 0$ 的三个根, 求 $(\alpha-1)(\beta-1)(\gamma-1)$ 的值。

If α , β , γ are the three roots of the equation $3x^3 + 12x^2 - 77x + 11 = 0$, find the value of $(\alpha-1)(\beta-1)(\gamma-1)$.

$$3x^3 + 12x^2 - 77x + 11 = 3(x-\alpha)(x-\beta)(x-\gamma)$$

Put $x=1$, we have

$$3(1-\alpha)(1-\beta)(1-\gamma) = 3+12-77+11$$

$$(1-\alpha)(1-\beta)(1-\gamma) = -17$$

$$(\alpha-1)(\beta-1)(\gamma-1) = 17$$

ANSWER: 【17】

第 21 至第 25 题，问答题，每题 6 分。

Question 21 to Question 25, short questions, each question carries 6 marks.

21. 有多少对整数 (x, y) 满足 $x^2 + y^2 \leq 300$?

[注：整数包括正、负整数及 0]

How many pairs of integers (x, y) are there that satisfy $x^2 + y^2 \leq 300$?

[Note: integers include positive, negative integers and 0]

$(0, 0)$ is a solution to $x^2 + y^2 \leq 300$.

Since $\lfloor \sqrt{300} \rfloor = 17$, there are $4 \times 17 = 68$ solutions with $x = 0$ or $y = 0$, but not both.

Now we consider the case $x > 0, y > 0$. There are

$\sum_{x=1}^{17} \lfloor \sqrt{300 - x^2} \rfloor$ solutions.

$$\begin{aligned} \sum_{x=1}^{17} \lfloor \sqrt{300 - x^2} \rfloor &= 3 \times 17 + 3 \times 16 + 2 \times 15 + 2 \times 14 + 13 + 12 + 11 + 10 + 8 + 6 + 3 \\ &= 220 \end{aligned}$$

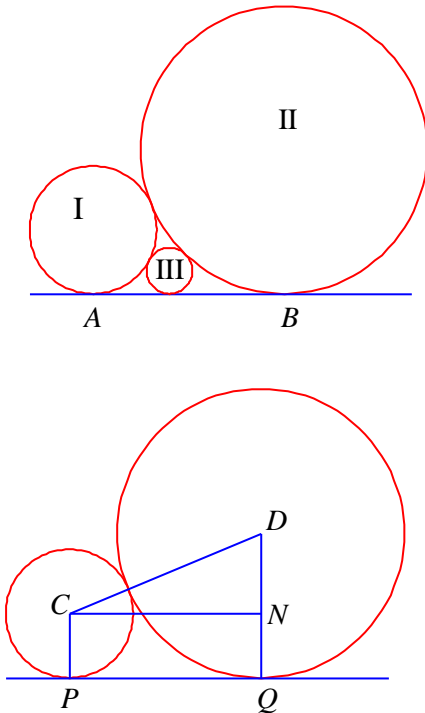
There are also 220 solutions for each of the cases where (a) $x > 0, y < 0$; (b) $x < 0, y > 0$; (c) $x < 0, y < 0$.

Hence, the total number of solutions is $880 + 68 + 1 = 949$.

ANSWER: 【949】

22. 下图中，三个圆两两外切，直线 AB 与三个圆都相切，圆 III 是最小的圆。若圆 I 与圆 II 的半径分别为 100 及 225，求圆 III 的半径。

In the figure below, the three circles are tangent externally to each other. The line AB is tangent to all three circles. Circle III is the smallest circle. If the radii of circles I and II are 100 and 225 respectively, find the radius of circle III.



We first look at this picture. If $PC = R_1$, $QD = R_2$, then $CD = R_1 + R_2$, $DN = R_2 - R_1$.

Hence, $PQ = CN = \sqrt{(R_1 + R_2)^2 - (R_2 - R_1)^2} = 2\sqrt{R_1 R_2}$.

If the radius of the circle III is R , and it is tangent to the line AB at the point C , then

$AC = 2\sqrt{RR_1}$, $CB = 2\sqrt{RR_2}$, $AB = 2\sqrt{R_1 R_2}$, where $R_1 = 100$, $R_2 = 225$.

Hence, $2\sqrt{RR_1} + 2\sqrt{RR_2} = 2\sqrt{R_1 R_2}$

$$\begin{aligned}\sqrt{R} &= \frac{\sqrt{R_1 R_2}}{\sqrt{R_1} + \sqrt{R_2}} \\ &= \frac{10 \times 15}{10 + 15} \\ &= 6 \\ R &= 36\end{aligned}$$

ANSWER: 【36】

23. 若 x 是实数, 求 $f(x) = \sqrt{2x-1} + \sqrt{243-2x}$ 的最大值。

If x is a real number, find the maximum value of $f(x) = \sqrt{2x-1} + \sqrt{243-2x}$.

Since $2ab \leq a^2 + b^2$, we have $a + b \leq \sqrt{2(a^2 + b^2)}$, and the equality holds if and only if $a = b$.

Hence,

$f(x) \leq \sqrt{2(2x-1+243-2x)} = 22$, and the equality holds if and only if $2x-1 = 243-2x$,

namely, $x = 61$.

The maximum value of $f(x) = \sqrt{2x-1} + \sqrt{243-2x}$ is 22.

ANSWER: 【22】

24. 已知 x, y, z 是正数且 $x + 2y + 3z = 21$ 。求 $2xy + 3xz + 6yz + 6xyz$ 的最大可能值。

Given that x, y, z are positive numbers such that $x + 2y + 3z = 21$. Find the largest possible value of $2xy + 3xz + 6yz + 6xyz$.

Notice that

$$(1+x)(1+2y)(1+3z) = 1 + (x+2y+3z) + (2xy+3xz+6yz) + 6xyz$$

By arithmetic mean-geometric mean inequality,

$$(1+x)(1+2y)(1+3z) \leq \left(\frac{1+x+1+2y+1+3z}{3} \right)^3$$

$$= 512$$

Equality holds when $x = 2y = 3z = 7$.

Hence, $2xy + 3xz + 6yz + 6xyz \leq 512 - 1 - 21 = 490$.

Namely, the largest possible value of $2xy + 3xz + 6yz + 6xyz$ is 490 and it is achieved when $x = 2y = 3z = 7$.

ANSWER: 【490】

25. 已知一数列 a_1, a_2, a_3, \dots 的定义为 $a_1 = 7$, 且对于 $n \geq 2$, $a_n = \frac{12a_{n-1}}{37 - a_{n-1}^2}$ 。

若 $S = a_1 + 2a_2 + 3a_3 + \dots + 2019a_{2019}$, 求 S 的各位数字之和。

Given that a_1, a_2, a_3, \dots is a sequence defined by $a_1 = 7$, and $a_n = \frac{12a_{n-1}}{37 - a_{n-1}^2}$ for all $n \geq 2$.

If $S = a_1 + 2a_2 + 3a_3 + \dots + 2019a_{2019}$, find the sum of the digits of S .

Notice that

$$a_2 = \frac{12 \times 7}{37 - 49} = -7$$

$$a_3 = \frac{12 \times (-7)}{37 - 49} = 7$$

Hence, $a_n = (-1)^{n-1} \times 7$.

$$\begin{aligned} S &= 7(1 - 2 + 3 - 4 + 5 - 6 + \dots + 2017 - 2018 + 2019) \\ &= 7(2019 - 1009) \\ &= 7 \times 1010 \\ &= 7070 \end{aligned}$$

Hence, the sum of the digits of S is 14.

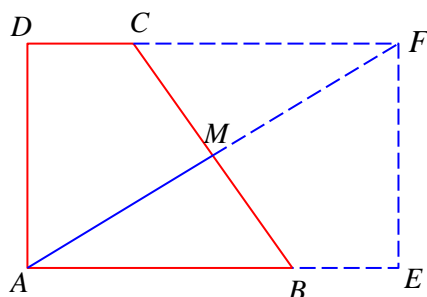
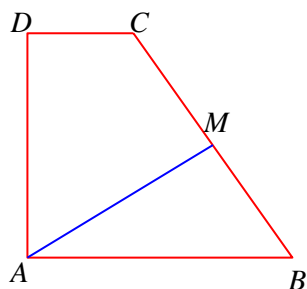
ANSWER: 【14】

第 26 至第 30 题，问答题，每题 8 分。

Question 26 to Question 30, short questions, each question carries 8 marks.

26. 下图中， $ABCD$ 是梯形， $AB \parallel CD$ ， $\angle DAB$ 是直角， M 是 BC 的中点。若 $AB + CD + AD = 62$ ， $AM = 25$ ，求梯形 $ABCD$ 的面积。

In the figure below, $ABCD$ is a trapezium with $AB \parallel CD$, $\angle DAB$ is a right angle, and M is the midpoint of BC . If $AB + CD + AD = 62$, $AM = 25$, find the area of trapezium $ABCD$.



Extend AB to the point E such that $BE = CD$, and extend DC to the point F such that $CF = AB$. Then $AEFD$ is a rectangle and $AM = \frac{1}{2}AF$.

Let $AE = 2x$, $AD = 2y$. Then

$$2x + 2y = 62$$

$$4x^2 + 4y^2 = 50^2$$

Hence, $x + y = 31$, $x^2 + y^2 = 625$

The area of trapezium $ABCD = 2xy = (x + y)^2 - (x^2 + y^2) = 961 - 625 = 336$.

ANSWER: 【336】

27. 设 k 是正整数。定义

$$(1+x)(1+2x)(1+3x)\dots(1+kx) = a_0 + a_1x + a_2x^2 + \dots + a_kx^k$$

并设 $N = a_0 + a_1 + a_2 + \dots + a_k$ 。若 N 可以被 2019 整除, 求 k 的最小可能值。

For a positive integer k , we write

$$(1+x)(1+2x)(1+3x)\dots(1+kx) = a_0 + a_1x + a_2x^2 + \dots + a_kx^k$$

Let $N = a_0 + a_1 + a_2 + \dots + a_k$. If N is divisible by 2019, find the smallest possible value of k .

Put $x=1$ in

$$(1+x)(1+2x)(1+3x)\dots(1+kx) = a_0 + a_1x + a_2x^2 + \dots + a_kx^k$$

we find that

$$\begin{aligned} N &= a_0 + a_1 + a_2 + \dots + a_k \\ &= 2 \times 3 \times 4 \times \dots \times (k+1) \\ &= (k+1)! \end{aligned}$$

Since the prime factorization of 2019 is 3×673 , $k+1 \geq 673$.

Therefore, the smallest possible value of k is 672.

ANSWER: 【672】

28. 已知集合 $S = \{1, 2, 3, \dots, 1000\}$ 包含由 1 到 1000 的整数。 S 的任何一个包含 N 个元素的子集一定包含两个相异的元素 a 与 b ，其和 $a+b$ 等于 1000。求 N 的最小可能值。

Given that the set $S = \{1, 2, 3, \dots, 1000\}$ contains all integers from 1 to 1000. Every subset of S that contains N elements must contain two distinct elements a and b such that their sum $a+b$ is 1000. Find the smallest possible value of N .

We claim that the smallest possible value of N is 502.

Consider the set $A = \{1, 2, 3, \dots, 499, 500, 1000\}$. It contains 501 elements with no two distinct elements sum to 1000.

Hence, if N is less than 502, we can find a subset of S with N elements which does not contain two distinct elements that sum to 1000. Just take any subset of A that contains N elements.

Now consider the following partition of the set S :

$$E_1 = \{1, 999\}, E_2 = \{2, 998\}, E_3 = \{3, 997\}, \dots, E_{499} = \{499, 501\}, E_{500} = \{500\}, \\ E_{501} = \{1000\}$$

By pigeonhole principle, any subset of S that contains 502 elements must contain at least 2 elements from one of the sets E_1, E_2, \dots, E_{499} , which has a sum of 1000.

Hence, the smallest possible value of N is 502.

ANSWER: 【502】

29. 已知

$$\frac{1}{19^{50}} = a_1 + \frac{a_2}{2!} + \frac{a_3}{3!} + \cdots + \frac{a_n}{n!}$$

其中 a_1, a_2, \dots, a_n 为非负的整数且对于所有 $2 \leq k \leq n$, $a_k \leq k-1$ 。求 n 的值。

Given that

$$\frac{1}{19^{50}} = a_1 + \frac{a_2}{2!} + \frac{a_3}{3!} + \cdots + \frac{a_n}{n!}$$

where a_1, a_2, \dots, a_n are nonnegative integers with $a_k \leq k-1$ for all $2 \leq k \leq n$. Find the value of n .

$$\text{Let } x = \frac{1}{19^{50}}.$$

Then

$$(n-1)! \times x = a_1(n-1)! + a_2 \frac{(n-1)!}{2!} + a_3 \frac{(n-1)!}{3!} + \cdots + a_{n-1} \frac{(n-1)!}{(n-1)!} + \frac{a_n}{n}$$

$$n! \times x = a_1 n! + a_2 \frac{n!}{2!} + a_3 \frac{n!}{3!} + \cdots + a_{n-1} \frac{n!}{(n-1)!} + a_n$$

These imply that $(n-1)! \times x$ might not be an integer but $n! \times x$ is. Hence, n is the smallest integer such that $19^{50} \mid n!$.

After determining n , a_n must be the remainder when $n! \times x$ is divided by n , a_{n-1} must be the remainder when $(n-1)! \times x - \frac{a_n}{n}$ is divided by $n-1$, and so on. The integers a_1, a_2, \dots, a_n are determined one-by-one in the reverse order.

Now let us determine the value of n . Since 19 is prime, we must have

$$\left\lfloor \frac{n}{19} \right\rfloor + \left\lfloor \frac{n}{19^2} \right\rfloor + \left\lfloor \frac{n}{19^3} \right\rfloor + \cdots = 50$$

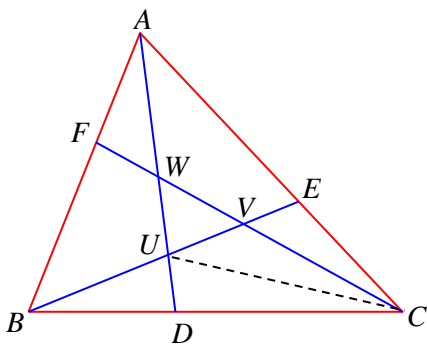
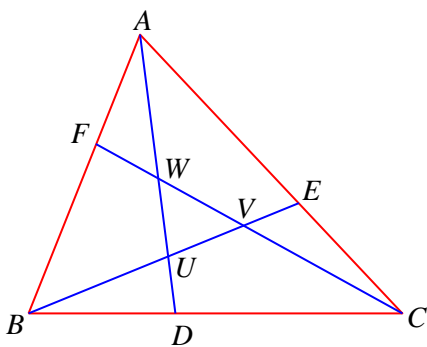
Since $\left\lfloor \frac{50}{19} \right\rfloor = 2$, we find that the smallest n satisfying this condition is

$$n = (50 - 2) \times 19 = 912$$

ANSWER: 【912】

30. 如下图所示, $\triangle ABC$ 中, D 、 E 、 F 分别是 BC 、 CA 及 AB 边上的点, 线段 AD 与线段 BE 相交于点 U , 线段 BE 与线段 CF 相交于点 V , 线段 CF 与线段 AD 相交于点 W , $BU = UE$, $CV = VF$, $AW = WD$ 。若 $\triangle ABC$ 的面积等于 12, $\triangle UVW$ 的面积为 $x - \sqrt{y}$, 其中 x 及 y 都是正整数, 且 y 不是平方数, 求 $x^2 + y$ 的值。

As shown in the figure below, in $\triangle ABC$, D , E , F are respectively points on BC , CA and AB . The line segment AD intersects the line segment BE at point U , the line segment BE intersects the line segment CF at point V , and the line segment CF intersects the line segment AD at point W . Given that $BU = UE$, $CV = VF$, $AW = WD$, and the area of $\triangle ABC$ is 12. If the area of $\triangle UVW$ is $x - \sqrt{y}$, where x and y are positive integers and y is not a perfect square, find the value of $x^2 + y$.



$$\text{Let } \frac{BD}{CD} = \alpha, \frac{CE}{EA} = \beta, \frac{AF}{FB} = \gamma.$$

If the area of $\triangle CUD$, denoted by $S_{\triangle CUD}$ is a , since $\frac{BD}{CD} = \alpha$, then $S_{\triangle BUD} = \alpha a$.

If $S_{\triangle AUB} = b$, since $BU = UE$, then $S_{\triangle AUE} = b$.

Since $BU = UE$, $S_{\triangle CUE} = S_{\triangle BUC} = (\alpha + 1)a$.

$$\text{Now, } \alpha = \frac{BD}{CD} = \frac{S_{\triangle ABD}}{S_{\triangle ACD}} = \frac{b + \alpha a}{b + (\alpha + 2)a}$$

$$\alpha b + \alpha(\alpha + 2)a = b + \alpha a$$

$$\alpha(\alpha + 1)a = (1 - \alpha)b$$

Hence,
$$\beta = \frac{CE}{EA} = \frac{S_{\triangle ACUE}}{S_{\triangle AUE}} = (\alpha + 1)\frac{a}{b} = \frac{1 - \alpha}{\alpha}$$

Using the same reasoning, we find that $\gamma = \frac{1 - \beta}{\beta}$, $\alpha = \frac{1 - \gamma}{\gamma}$.

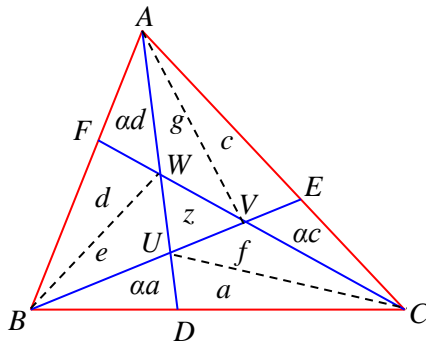
This gives
$$\gamma = \frac{2\alpha - 1}{1 - \alpha}, \alpha = \frac{2 - 3\alpha}{2\alpha - 1}$$

From this, it follows that $2\alpha^2 + 2\alpha - 2 = 0$

$$\alpha = \frac{-1 + \sqrt{5}}{2}$$

and we find that $\beta = \gamma = \alpha$.

To find the area of $\triangle UVW$, we denote the areas of each region in the triangle as shown below:



Using $BU = UE$, $CV = VF$ and $AW = WD$, we have

$$\alpha a + a = f + \alpha c, \quad \alpha d + d + e = c + g + z$$

$$\alpha c + c = g + \alpha d, \quad \alpha a + a + f = d + e + z$$

$$\alpha d + d = e + \alpha a, \quad \alpha c + c + g = a + f + z$$

The sum of the three formulas on the left gives

$$e + f + g = a + c + d$$

The sum of the three formulas on the right gives

$$z = \frac{\alpha(a + c + d)}{3}$$

Now, the area of $\triangle ABC$ is 12. Hence,

$$12 = \alpha(a + c + d) + (a + c + d) + e + f + g + z$$

$$\left(\alpha + 2 + \frac{\alpha}{3}\right)(a + c + d) = 12$$

$$\begin{aligned}z &= \frac{\alpha}{3} \times \frac{3}{4\alpha+6} \times 12 \\&= 3 \times \frac{\sqrt{5}-1}{\sqrt{5}+2} \\&= 3(7-3\sqrt{5}) \\&= 21 - \sqrt{405}\end{aligned}$$

Hence, $x = 21$, $y = 405$, $x^2 + y = 441 + 405 = 846$

ANSWER: 【846】