厦门大学马来西亚分校
陈景润杯中学数学比赛

CHEN JINGRUN'S CUP SECONDARY SCHOOL MATHEMATICS COMPETITION 2019

**** 高阶组 ****

SENIOR CATEGORY

日期：2019 年 4 月 9 日　　　　时间：上午 10 时至中午 12 时

Date: 9th April 2019　　　　Time: 10:00 a.m. to 12:00 p.m.

考生须知

Instructions and Information

1. 本试卷共有 30 题。
   This paper contains 30 questions.
   - 第 1 题至第 10 题，选择题， 每题 4 分。
     Question 1 to Question 10, multiple choice questions, each question carries 4 marks.
   - 第 11 题至第 30 题，问答题，每题的答案是一个介于 0 至 1000 之间的整数。
     Question 11 to Question 30, short questions. For each question, the answer is an integer between 0 and 1000.
     - ➢ 第 11 题至第 20 题每题 5 分。
       Question 11 to Question 20, each question carries 5 marks.
     - ➢ 第 21 题至第 25 题每题 6 分。
       Question 21 to Question 25, each question carries 6 marks.
     - ➢ 第 26 题至第 30 题每题 8 分。
       Question 26 to Question 30, each question carries 8 marks.
2. 请在答案纸内适当的空格中用 2B 铅笔清楚的写出每题的答案。对于选择题，必须填写 A，B，C，D 或 E 作为答案。每题只能填入一个答案，否则以答错论。
   Please use 2B pencils to write your answers in the appropriate boxes provided on the answer sheet. For each multiple choice question, please write A, B, C, D or E as answer. If more than one answer is found for a question, no credits would be given for that question.
3. 所有的图形并没有按照比例作图，只作为辅助之用。
   All the diagrams are not drawn to scale. They are intended as aids only.
4. 不许使用计算器，数学工具，手机或其他计算器。
   No calculators, maths stencils, mobile phones or other calculating aids are permitted.
5. 在答案纸上清楚写上姓名，考生编号，学校名称及就读年级。
   Write your name, candidate number, name of school and year of study clearly on the answer sheet.
6. 在监考老师宣布比赛开始之后，才可以翻开此考卷开始作答。
   Do not open this question booklet until you are told to do so.

~~ 说明 ~~

~~ Notes ~~

在这份试卷中，$\lfloor x \rfloor$表示小于或等于x的最大整数。

例如：$\lfloor 2 \rfloor = 2$，$\lfloor -2 \rfloor = -2$，$\lfloor 2.6 \rfloor = 2$，$\lfloor -2.6 \rfloor = -3$。

In this paper, $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

For example, $\lfloor 2 \rfloor = 2$, $\lfloor -2 \rfloor = -2$, $\lfloor 2.6 \rfloor = 2$, $\lfloor -2.6 \rfloor = -3$.

第 1 至第 10 题，选择题，每题 4 分。

Question 1 to Question 10, multiple choice questions, each question carries 4 marks.

1. 已知多项式 $p(x)$除以$2x^2-3x-2$的余式是$4x-7$，求 $p(x)$除以$2x+1$的余数。

   Given that the polynomial $p(x)$ leaves a remainder of $4x-7$ when it is divided by $2x^2-3x-2$, find the remainder when $p(x)$ is divided by $2x+1$.

   A. 9　　B. 5　　C. –5　　D. –8　　E. –9

2. 一个箱子中有 9 粒白球，7 粒红球。任意从箱子中取出一球给林慧，再取出一球给李明。求李明拿到的球是红球的概率。

   There are 9 white balls and 7 red balls in a box. A ball is randomly taken out from the box and passed to LinHui, then a second ball is taken out and passed to LiMing. Find the probability that the ball obtained by LiMing is a red ball.

   A. $\frac{7}{16}$　　B. $\frac{9}{16}$　　C. $\frac{2}{5}$　　D. $\frac{8}{15}$　　E. $\frac{7}{15}$

3. 波波，东东，卡卡三人一起去超市。波波买了一粒苹果，两粒橙及三粒芒果，花了 RM17.60。东东买了两粒苹果，三粒橙，一粒芒果，花了 RM12.00。卡卡买了三粒苹果，一粒橙，两粒芒果，花了 RM14.20。问一粒苹果的单价是多少?

   Bobo, Dongdong, Kaka went to a grocery store together. Bobo spent RM17.60 to buy an apple, two oranges and three mangos. Dongdong spent RM12.00 to buy two apples, three oranges and one mango. Kaka spent RM14.20 to buy three apples, one orange and two mangos. What is the unit price of an apple?

   A. RM1.30　　B. RM1.40　　C. RM1.50　　D. RM1.60　　E. RM1.70

4. 当一粒球的半径增加了10%，求它的体积增加的百分比。
When the radius of a ball is increased by 10%, find the percentage increase in its volume.

A. 20% B. 21% C. 30% D. 33% E. 33.1%

5. 求$\sqrt[3]{4035^3-2017^3-3\times4035\times2017\times2018}$。
Find $\sqrt[3]{4035^3-2017^3-3\times4035\times2017\times2018}$.

A. 2017 B. 2018 C. 2019 D. 2020 E. 2021

6. 已知组合数$\binom{n}{k}=\frac{n!}{k!(n-k)!}$是由 n 件不同的物品选出k件的方法数。化简

$$\binom{37}{1}-\binom{37}{2}+\binom{37}{3}-\binom{37}{4}+\cdots+\binom{37}{17}-\binom{37}{18}$$

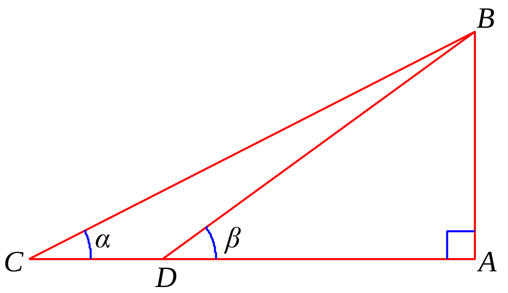
Given that the combination number $\binom{n}{k}=\frac{n!}{k!(n-k)!}$ is the number of ways to choose k objects from n different objects. Simplify

$$\binom{37}{1}-\binom{37}{2}+\binom{37}{3}-\binom{37}{4}+\cdots+\binom{37}{17}-\binom{37}{18}$$

A. $\binom{36}{18}$ B. $-\binom{36}{18}$ C. $1+\binom{36}{18}$ D. $1-\binom{36}{18}$ E. $\binom{36}{18}-1$

7. 下图中，ΔABC 是直角三角形，$\angle A$ 是直角，D 是 AC 上的一点。$\angle BCA=\alpha$，$\angle BDA=\beta$。若 $CD=d$，$AB=h$，则

In the figure below, ΔABC is a right-angled triangle with $\angle A$ being the right angle. D is a point on AC, and $\angle BCA=\alpha$, $\angle BDA=\beta$. If $CD=d$, $AB=h$, then



A. $d=h(\sin\beta-\sin\alpha)$

B. $d=h(\tan\beta-\tan\alpha)$

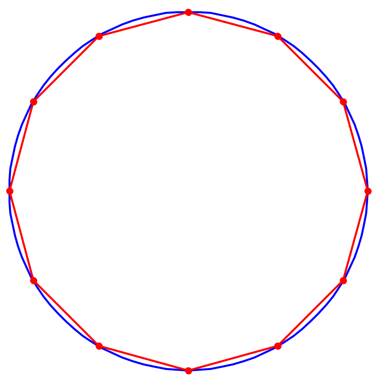
C. $d=h(\cot\alpha-\cot\beta)$

D. $d=h(\cos\alpha-\cos\beta)$

E. $d=h(\sec\beta-\sec\alpha)$

8. 下图所示为一圆与一内接于圆中的正十二边形。若圆的半径为 $\sqrt{6}+\sqrt{2}$，求正十二边形的周长。

The figure below shows a circle and an inscribed regular 12-gon. If the radius of the circle is $\sqrt{6}+\sqrt{2}$, find the perimeter of the regular 12-gon.



A. 12 B. 18 C. 24 D. 36 E. 48

9. 已知 n 是整数且 $n\left(\sqrt{399}-\sqrt{397}\right)<1$。求 n 的最大可能值。

Given that n is an integer such that $n\left(\sqrt{399}-\sqrt{397}\right)<1$. Find the largest possible value of n.

A. 18 B. 19 C. 38 D. 39 E. 40

10. 将一个边长为 13 cm 的立方体的表面涂上红色，再将它切割成边长为 1 cm 的小立方体。在这些小立方体中，有多少个恰好有两面被涂上红色?

After the faces of a cube with side length 13 cm are colored red, the cube is dissected into small cubes, each with side length 1 cm. Among the small cubes, how many of them have exactly two red faces?

A. 169 B. 156 C. 144 D. 132 E. 121

第 11 至第 20 题，问答题，每题 5 分。

Question 11 to Question 20, short questions, each question carries 5 marks.

11. 已知131个数 x_1， x_2， $\cdots$， x_{131} 的平均数是 131。对于 $1\le k\le 131$， $y_k=x_k+k$，求 y_1， y_2， $\cdots$， y_{131} 这131个数的平均数。

Given that the average of the 131 numbers x_1, x_2, $\cdots$, x_{131} is 131. If $y_k=x_k+k$ for $1\le k\le 131$, find the average of the 131 numbers y_1, y_2, $\cdots$, y_{131}.

12. 求满足不等式 $\log_{\frac{1}{2}}\left(5n-43\right)+8\ge 0$ 的最大整数 n。

Find the largest integer n that satisfies the inequality $\log_{\frac{1}{2}}\left(5n-43\right)+8\ge 0$.

13. 有一群人参加会议。第一阶段结束后，66位女士离开会场，使得会场中所剩下的男、女之比例为$5:3$。第二阶段结束后，又有 132 位男士离开会场，使得会场中所剩下的男、女之比例为$3:4$。问会议刚开始时有多少位女士在会场?

A group of people participated in a conference. After the first session, 66 of the female participants left, yielding a ratio of male to female participants in the conference room to be $5:3$. After the second session, another 132 male participants left, yielding a ratio of male to female participants in the conference room to be $3:4$. How many female participants were there at the beginning of the conference?

14. 已知圆$x^2+y^2+4x-18y-60=0$与直线$x-2y=5$相交于点A及点B，点C是圆的圆心。求ΔABC的面积。

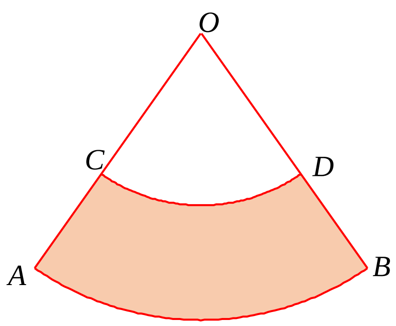
Given that the circle $x^2+y^2+4x-18y-60=0$ intersects the line $x-2y=5$ at points A and B, and C is the center of the circle. Find the area of ΔABC.

15. 已知$N=2^2\times3\times5^2\times6^3\times7^4$。求$N$的正因子的个数。

Given that $N=2^2\times3\times5^2\times6^3\times7^4$. Find the number of positive factors of N.

16. 下图中，OAB及OCD是扇形。若$\overset{\frown}{AB}=20\pi$，$\overset{\frown}{CD}=12\pi$，$AC=14$，阴影部分的面积为$S\pi$，求$S$的值。

In the figure below, OAB and OCD are circular sectors. If $\overset{\frown}{AB}=20\pi$, $\overset{\frown}{CD}=12\pi$, $AC=14$, and the area of the shaded region is $S\pi$, find the value of S.



17. 已知 $A=\{1,2,3,4\}$， $B=\{5,6,7,8,9,10,11\}$ 。有多少个由 A 到 B 的一对一函数?

Given that $A=\{1,2,3,4\}$, $B=\{5,6,7,8,9,10,11\}$. How many one-to-one functions are there from A to B?

18. 求函数 $f(x)=303-\dfrac{200x^2}{x^4+16}$ 的最小值。

Find the minimum value of the function $f(x)=303-\dfrac{200x^2}{x^4+16}$.

19. 已知 x 是一整数且 $\left\lfloor\dfrac{x-128}{7}\right\rfloor=-3$，求 x 的最大可能值。

Given that x is an integer and $\left\lfloor\dfrac{x-128}{7}\right\rfloor=-3$, find the largest possible value of x.

20. 若 α， β， γ 是方程式 $3x^3+12x^2-77x+11=0$ 的三个根，求 $(\alpha-1)(\beta-1)(\gamma-1)$ 的值。

If α, β, γ are the three roots of the equation $3x^3+12x^2-77x+11=0$, find the value of $(\alpha-1)(\beta-1)(\gamma-1)$.

第 21 至第 25 题，问答题，每题 6 分。

Question 21 to Question 25, short questions, each question carries 6 marks.

21. 有多少对整数 (x, y) 满足 $x^2+y^2\le 300$？

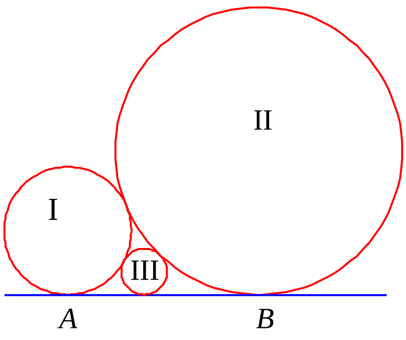
[注：整数包括正、负整数及 0]

How many pairs of integers (x, y) are there that satisfy $x^2+y^2\le 300$?

[Note: integers include positive, negative integers and 0]

22. 下图中，三个圆两两外切，直线 AB 与三个圆都相切，圆 III 是最小的圆。若圆 I 与圆 II 的半径分别为 100 及 225，求圆 III 的半径。

In the figure below, the three circles are tangent externally to each other. The line AB is tangent to all three circles. Circle III is the smallest circle. If the radii of circles I and II are 100 and 225 respectively, find the radius of circle III.



23. 若 x 是实数，求 $f(x)=\sqrt{2x-1}+\sqrt{243-2x}$ 的最大值。

If x is a real number, find the maximum value of $f(x)=\sqrt{2x-1}+\sqrt{243-2x}$.

24. 已知 x， y， z 是正数且 $x+2y+3z=21$。求 $2xy+3xz+6yz+6xyz$ 的最大可能值。

Given that x, y, z are positive numbers such that $x+2y+3z=21$. Find the largest possible value of $2xy+3xz+6yz+6xyz$.

25. 已知一数列 a_1， a_2， a_3， $\cdots$ 的定义为 $a_1=7$，且对于 $n\geq 2$， $a_n=\dfrac{12a_{n-1}}{37-a_{n-1}^2}$。

若 $S=a_1+2a_2+3a_3+\cdots+2019a_{2019}$，求 S 的各位数字之和。

Given that a_1, a_2, a_3, $\cdots$ is a sequence defined by $a_1=7$, and $a_n=\dfrac{12a_{n-1}}{37-a_{n-1}^2}$ for all $n\geq 2$.

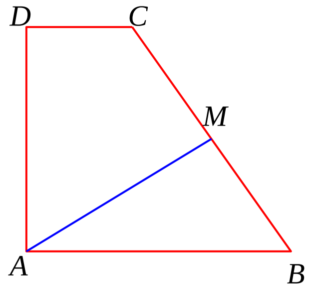
If $S=a_1+2a_2+3a_3+\cdots+2019a_{2019}$, find the sum of the digits of S.

第 26 至第 30 题，问答题，每题 8 分。

Question 26 to Question 30, short questions, each question carries 8 marks.

26. 下图中，$ABCD$ 是梯形，$AB//CD$，$\angle DAB$ 是直角，M 是 BC 的中点。若 $AB+CD+AD=62$，$AM=25$，求梯形 $ABCD$ 的面积。

In the figure below, $ABCD$ is a trapezium with $AB//CD$, $\angle DAB$ is a right angle, and M is the midpoint of BC. If $AB+CD+AD=62$, $AM=25$, find the area of trapezium $ABCD$.



27. 设 k 是正整数。定义

$$(1+x)(1+2x)(1+3x)\ldots(1+kx)=a_0+a_1x+a_2x^2+\cdots+a_kx^k$$

并设 $N=a_0+a_1+a_2+\cdots+a_k$。若 N 可以被 2019 整除，求 k 的最小可能值。

For a positive integer k, we write

$$(1+x)(1+2x)(1+3x)\ldots(1+kx)=a_0+a_1x+a_2x^2+\cdots+a_kx^k$$

Let $N=a_0+a_1+a_2+\cdots+a_k$. If N is divisible by 2019, find the smallest possible value of k.

28. 已知集合 $S=\{1,2,3,\cdots,1000\}$ 包含由 1 到 1000 的整数。S 的任何一个包含 N 个元素的子集合一定包含两个相异的元素 a 与 b，其和 $a+b$ 等于 1000。求 N 的最小可能值。

Given that the set $S=\{1,2,3,\cdots,1000\}$ contains all integers from 1 to 1000. Every subset of S that contains N elements must contain two distinct elements a and b such that their sum $a+b$ is 1000. Find the smallest possible value of N.

29. 已知

$$\frac{1}{19^{50}}=a_1+\frac{a_2}{2!}+\frac{a_3}{3!}+\cdots+\frac{a_n}{n!}$$

其中 a_1， a_2， $\cdots$， a_n 为非负的整数且对于所有 $2\le k\le n$， $a_k\le k-1$。求 n 的值。

Given that

$$\frac{1}{19^{50}}=a_1+\frac{a_2}{2!}+\frac{a_3}{3!}+\cdots+\frac{a_n}{n!}$$

where a_1, a_2, $\cdots$, a_n are nonnegative integers with $a_k\le k-1$ for all $2\le k\le n$. Find the value of n.

30. 如下图所示，ΔABC 中， D 、 E 、 F 分别是 BC 、 CA 及 AB 边上的点，线段 AD 与线段 BE 相交于点 U ，线段 BE 与线段 CF 相交于点 V ，线段 CF 与线段 AD 相交于点 W ， $BU=UE$ ， $CV=VF$ ， $AW=WD$ 。若 ΔABC 的面积等于 12， ΔUVW 的面积为 $x-\sqrt{y}$ ，其中 x 及 y 都是正整数，且 y 不是平方数，求 x^2+y 的值。

As shown in the figure below, in ΔABC, D, E, F are respectively points on BC, CA and AB. The line segment AD intersects the line segment BE at point U, the line segment BE intersects the line segment CF at point V, and the line segment CF intersects the line segment AD at point W. Given that $BU=UE$, $CV=VF$, $AW=WD$, and the area of ΔABC is 12. If the area of ΔUVW is $x-\sqrt{y}$, where x and y are positive integers and y is not a perfect square, find the value of x^2+y.