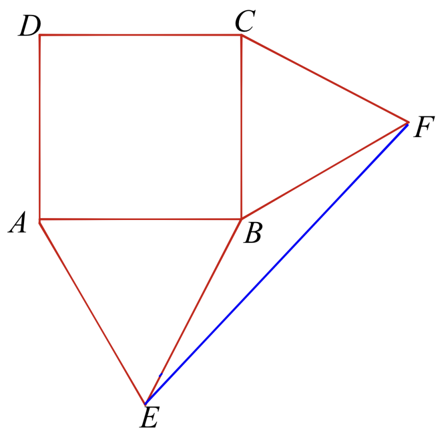
Question S-01 [4 points]

In the figure shown below, $ABCD$ is a rectangle with $AB = 7$ and $BC = 4\sqrt{3}$. ΔABE and ΔBCF are equilateral triangles. If $x = EF$, find the value of x^2.

下图中，$ABCD$ 是长方形，$AB = 7$，$BC = 4\sqrt{3}$。ΔABE 及 ΔBCF 是等边三角形。若 $x = EF$，求 x^2 的值。



Answer: [181]

Solutions:

$\angle EBF = 150^\circ$. By cosine rule

$$x^2 = 7^2 + (4\sqrt{3})^2 - 2 \times 7 \times 4\sqrt{3} \times \cos 150^\circ = 181.$$

Question S-02 [4 points]

There are 6 blue pens and 4 red pens in a box. After Ms Zhang randomly takes away two pens from the box, Ms Xiao randomly takes away another two. If the probability that the two pens Ms Xiao takes are both red is p, find the value of $5040p$.

一个盒子中有 6 支蓝笔，4 支红笔。张老师先任意从盒子中取走两支笔，萧老师再任意取两支。若萧老师拿到的两支都是红笔的概率是 p，求 $5040p$ 的值。

Answer: [672]

Solutions:

$$p = \frac{4 \times 3}{10 \times 9}.$$

Hence, $5040p = 672$.

Question S-03 [4 points]

Let

$$A_1 = \left\{\log_{2k-1}(2k+1) \mid k \text{ is an integer},\ 2 \leq k \leq 1093\right\},$$
$$A_2 = \left\{\log_{2k+1}(2k-1) \mid k \text{ is an integer},\ 2 \leq k \leq 364\right\}.$$

Define P_1 to be the product of all the elements in A_1, and P_2 the product of all the elements in A_2. Find $\frac{P_1}{P_2}$.

设

$$A_1 = \left\{\log_{2k-1}(2k+1) \mid k \text{ 是整数},\ 2 \leq k \leq 1093\right\},$$
$$A_2 = \left\{\log_{2k+1}(2k-1) \mid k \text{ 是整数},\ 2 \leq k \leq 364\right\}$$

定义 P_1 为 A_1 中所有元素的乘积，P_2 为 A_2 中所有元素的乘积。求 $\frac{P_1}{P_2}$。

Answer: [42]

Solutions:

If $k = 364$, $2k+1 = 729 = 3^6$.

If $k = 1093$, $2k+1 = 2187 = 3^7$.

Then

$$P_1 = \frac{\log 5}{\log 3} \times \frac{\log 7}{\log 5} \times \dots \frac{\log 2187}{\log 2185} = \frac{\log 3^7}{\log 3} = 7,$$
$$P_2 = \frac{\log 3}{\log 5} \times \frac{\log 5}{\log 7} \times \dots \frac{\log 727}{\log 729} = \frac{\log 3}{\log 3^6} = \frac{1}{6}.$$

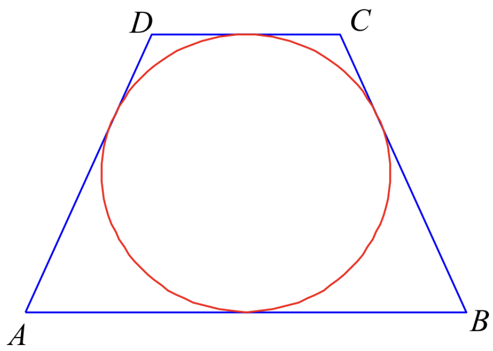
Hence,

$$\frac{P_1}{P_2} = 42.$$

Question S-04 [4 points]

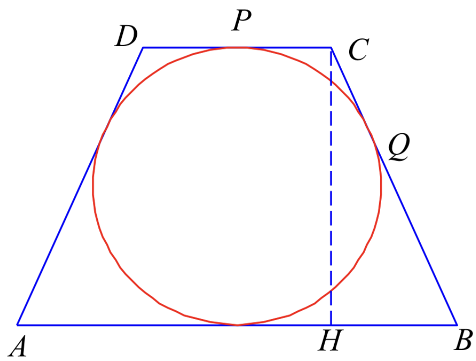
As shown in the figure below, a circle is inscribed in the trapezium $ABCD$. Given that $AB = 50$, $CD = 18$ and $AD = BC$. If the area of the circle is πa, find the value of a.

如下图所示，一圆内切于梯形 $ABCD$ 中。已知 $AB = 50$，$CD = 18$ 且 $AD = BC$。若圆的面积为 πa，求 a 的值。



Answer: [225]

Solutions:



$$CQ = CP = \frac{1}{2}CD = 9,$$
$$BQ = \frac{1}{2}AB = 25$$

Hence, $BC = 34$, $BH = 16$. This gives $2r = CH = 30$, where r is the radius of the circle. Hence $\pi a = 225\pi$, and so $a = 225$.

Question S-05 [4 points]

The three side lengths of an isosceles triangle are integers. If the longest side has length 1000, the shortest side has length x, find the smallest possible value of x.

一等腰三角形三边的长都是整数。若最长的边长1000，最短的边长 x，求 x 的最小可能值。

Answer: [1]

Solutions:

The smallest possible value of x is 1, when the two equal sides both have length 1000.

Question S-06 [4 points]

If k is a nonnegative integer and n is a positive integer such that $k \leq n$, define $\binom{n}{k}$ to be the number of ways to choose k objects from n objects. Find the value of n such that

$$\sum_{k=1}^{n} \binom{2n+1}{k} = 2^{586} - 1.$$

当 k 是非负的整数，n 是正整数且 $k \leq n$ 时，定义 $\binom{n}{k}$ 为从 n 个不同的东西选取 k 个的方法数。求 n 的值使得

$$\sum_{k=1}^{n} \binom{2n+1}{k} = 2^{586} - 1$$

Answer: [293]

Solutions:

Since

$$\binom{2n+1}{k} = \binom{2n+1}{2n+1-k},$$

we find that

$$\sum_{k=1}^{n} \binom{2n+1}{k} = \frac{1}{2} \sum_{k=1}^{2n} \binom{2n+1}{k} = \frac{1}{2} \left(2^{2n+1} - 2\right) = 2^{2n} - 1$$

Hence, $2n = 586$ and $n = 293$.

Question S-07 [4 points]

Given that the two curves $y = x^2 - 2x - k$ and $y = -2x^2 + 34x - 2k + 1$ are tangent to each other, find the value of k.

已知曲线 $y = x^2 - 2x - k$ 和曲线 $y = -2x^2 + 34x - 2k + 1$ 相切，求 k 的值。

Answer: [109]

Solutions:

Solving $y = x^2 - 2x - k$ and $y = -2x^2 + 34x - 2k + 1$ for intersection point, we get

$$3x^2 - 36x + k - 1 = 0.$$

There should be only one real solution. Hence,

$$36^2 - 12(k - 1) = 0.$$

This gives $k = 109$.

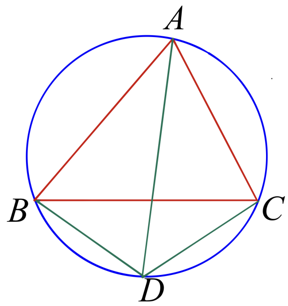
Question S-08 [4 points]

In ΔABC, $\angle BAC = 60°$. D is a point on the circumcircle of ΔABC such that AD bisects $\angle BAC$. If $BC^2 = 789$, find BD^2.

ΔABC 中，$\angle BAC = 60°$。D 是 ΔABC 外接圆上的一点使得 AD 平分 $\angle BAC$。若 $BC^2 = 789$，求 BD^2。

Answer: [263]

Solutions:



$$\angle BCD = \angle BAD = \angle CAD = \angle CBD = 30°.$$

Hence ΔBCD is an isosceles triangle with internal angles $120°$, $30°$ and $30°$.

This gives

$$BD^2 = \frac{BC^2}{3} = 263.$$

Question S-09 [4 points]

Given that (a, b) is the point on the curve $x^2 + y^2 - 44x + 38y + 556 = 0$ that is closest to the line $15x - 8y + 56 = 0$. Find $a^2 + b^2$.

已知 (a, b) 是在曲线 $x^2 + y^2 - 44x + 38y + 556 = 0$ 上与直线 $15x - 8y + 56 = 0$ 最靠近的点，求 $a^2 + b^2$。

Answer: [170]

Solutions:

$$x^2 + y^2 - 44x + 38y + 556 = (x - 22)^2 + (y + 19)^2 - 17^2.$$

Hence, $x^2 + y^2 - 44x + 38y + 556 = 0$ is a circle with center at $(22, -19)$ and with radius 17.
The line from (a, b) to $(22, -19)$ must be perpendicular to the line $15x - 8y + 56 = 0$. Hence, it has equation $8x + 15y + 109 = 0$.
The point (a, b) is the intersection point of $8x+15y+109 = 0$ and $x^2+y^2-44x+38y+556 = 0$ that is closer to $15x - 8y + 56 = 0$.
Solving the equations, we find that $(a, b) = (7, -11)$, and hence, $a^2 + b^2 = 170$.

Question S-10 [4 points]

Given that x is a positive integer such that

$$\sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+\ldots}}}}=31.$$

Find x.

已知 x 是一正整数使得

$$\sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+\ldots}}}}=31$$

求 x。

Answer: [930]

Solutions:

We have

$$\sqrt{x+31}=31.$$

Hence,

$$x=31^2-31=930.$$

Question S-11 [5 points]

How many ways are there to distribute 20 identical pens to 5 teachers so that every teacher gets at least two pens?

有多少种方法可以将 20 支一样的笔分给 5 位老师，每人至少分两支?

Answer: [1001]

Solutions:

Let $x_1 + 2$, $x_2 + 2$, $x_3 + 2$, $x_4 + 2$ and $x_5 + 2$ be respectively the number of pencils given to the teachers. x_1, x_2, x_3, x_4 and x_5 are nonnegative integers with

$$x_1 + x_2 + x_3 + x_4 + x_5 = 10.$$

The number of solutions is

$$\binom{10+4}{4} = 1001.$$

Question S-12 [5 points]

A school has four clubs, A, B, C, D, whose members are students in this school. Every two clubs have 227 common members. Every three clubs have 117 common members. There are exactly 17 students that join all four clubs. At least how many students does club A have?

已知一间学校有 A、B、C、D 四个学会，每个学会的会员都是该校的学生。每两个学会有 227 位共同会员，每三个学会有 117 位共同会员，恰有 17 位学生是这四个学会的共同会员。学会 A 至少有几位会员？

Answer: [347]

Solutions:

Let A_1, A_2, A_3, A_4 be respectively the set of students in the club A, club B, club C and club D.

$$\begin{aligned} n(A_1 \cap (A_2 \cup A_3 \cup A_4)) =& n(A_1 \cap A_2) + n(A_1 \cap A_3) + n(A_1 \cap A_4) \\ & - n(A_1 \cap A_2 \cap A_3) - n(A_1 \cap A_2 \cap A_4) - n(A_1 \cap A_3 \cap A_4) \\ & + n(A_1 \cap A_2 \cap A_3 \cap A_4) \\ =& 3 \times 227 - 3 \times 117 + 17 = 347 \end{aligned}$$

Hence, club A has at least 347 students.

Question S-13 [5 points]

Given that $\sum_{k=1}^{n} \frac{1}{\sqrt{3k+1}+\sqrt{3k-2}} = 17$, find the value of n.

已知 $\sum_{k=1}^{n} \frac{1}{\sqrt{3k+1}+\sqrt{3k-2}} = 17$，求 n 的值。

Answer: [901]

Solutions:

$$\sum_{k=1}^{n} \frac{1}{\sqrt{3k+1}+\sqrt{3k-2}} = \sum_{k=1}^{n} \frac{\sqrt{3k+1}-\sqrt{3k-2}}{3} = \frac{1}{3}\left(\sqrt{3n+1}-1\right)$$

This gives $\sqrt{3n+1} = 52$ and hence $n = 901$.

Question S-14 [5 points]

Given that when the polynomial $f(x)$ is divided by $(x-1)(x-2)(x-3)(x-4)$, the remainder is $x(x-1)(x-2)$. When $f(x)$ is divided by $(x-2)(x-3)$, the remainder is $g(x)$. Find the remainder when $g(x)$ is divided by $x-4$.

已知多项式 $f(x)$ 除以 $(x-1)(x-2)(x-3)(x-4)$ 得余项 $x(x-1)(x-2)$，$f(x)$ 除以 $(x-2)(x-3)$ 得余项 $g(x)$。求 $g(x)$ 除以 $x-4$ 的余数。

Answer: [12]

Solutions:

$$\begin{aligned} f(x) =& q_1(x)(x-1)(x-2)(x-3)(x-4) + x(x-1)(x-2) \\ =& q_2(x)(x-2)(x-3) + g(x) \end{aligned}$$

Let $g(x) = ax + b$. Then $2a + b = g(2) = f(2) = 0$, $3a + b = g(3) = f(3) = 6$.

Hence, $a = 6$, $b = -12$.

When $g(x)$ is divided by $x - 4$, the remainder is $g(4) = 4a + b = 12$.

Question S-15 [5 points]

Given that a is an integer and the maximum value of $Q(x)=\dfrac{a-511x-146x^2}{2x^2+7x+10}$ is 167, find the value of a.

已知 a 是一整数且 $Q(x)=\dfrac{a-511x-146x^2}{2x^2+7x+10}$ 的最大值为 167，求 a 的值。

Answer: [200]

Solutions:

$$\begin{aligned}Q(x)&=\frac{a-511x-146x^2}{2x^2+7x+10}\\&=\frac{a+730}{2x^2+7x+10}-73\\&=\frac{a+730}{2\left(x+\frac{7}{4}\right)^2+\frac{31}{8}}-73\end{aligned}$$

If $a\le -730$, $Q(x)<0$.

Since the maximum value of $Q(x)$ is 167, $a>-730$.

When $x=-\frac{7}{4}$, $Q(x)$ has maximum value

$$\frac{8}{31}(a+730)-73=167.$$

This gives $a=200$.

Question S-16 [5 points]

Find the smallest positive integer n such that $\sqrt{n+2020}-\sqrt{n}<20$.

求最小的正整数 n 使得 $\sqrt{n+2020}-\sqrt{n}<20$。

Answer: [1641]

Solutions:

$$\sqrt{n+2020}-\sqrt{n}<20$$
$$\frac{2020}{\sqrt{n+2020}+\sqrt{n}}<20$$
$$\sqrt{n+2020}+\sqrt{n}>101$$

We first consider $\sqrt{n+2020}+\sqrt{n}=101$. Then $\sqrt{n+2020}-\sqrt{n}=20$.
Hence, $\sqrt{n}=\frac{81}{2}=40+\frac{1}{2}$ and $n=1600+40+\frac{1}{4}$.

$$\sqrt{1640+2020}-\sqrt{1640}=20.001$$
$$\sqrt{1641+2020}-\sqrt{1641}=19.9969$$

For $n\leq 1640$, $\sqrt{n+2020}-\sqrt{n}\geq\sqrt{1640+2020}-\sqrt{1640}>20$.
Therefore, the smallest positive integer n such that $\sqrt{n+2020}-\sqrt{n}<20$ is 1641.

Question S-17 [5 points]

How many real solutions does the equation $x + \sqrt{x^2 + \sqrt{x^3 + 16}} = 2$ have?

方程式 $x + \sqrt{x^2 + \sqrt{x^3 + 16}} = 2$ 有几个实数解?

Answer: [1]

Solutions:

$$\sqrt{x^2 + \sqrt{x^3 + 16}} = 2 - x$$
$$x^2 + \sqrt{x^3 + 16} = 4 - 4x + x^2$$
$$\sqrt{x^3 + 16} = 4 - 4x$$
$$x^3 + 16 = 16 - 32x + 16x^2$$
$$x(x^2 - 16x + 32) = 0$$
$$x = 0 \quad \text{or} \quad x = 8 \pm 4\sqrt{2}$$

Notice that $8 \pm 4\sqrt{2} > 0$.
If $x > 0$, $x + \sqrt{x^2 + \sqrt{x^3 + 16}} > 2$.
Hence, $x + \sqrt{x^2 + \sqrt{x^3 + 16}} = 2$ only has one solution $x = 0$.

Question S-18 [5 points]

A line separates the plane into 2 regions. Now the plane is separated into N regions by 41 distinct lines. If the maximum value of N is M, and the minimum value of N is m, find $M + m$.

一条直线将平面分成两个区域。现平面被 41 条相异的直线分成 N 个区域。若 N 的最大值与最小值分别为 M 与 m, 求 $M + m$ 的值。

Answer: [904]

Solutions:

Let M_k be the maximum number of regions on the plane separated by k lines. When we add in the $k + 1$ line, it can add $k + 1$ regions. Therefore,

$$M_{k+1} - M_k = k + 1.$$

This shows that

$$M_{41} - M_1 = 41 + 40 + \ldots + 2.$$

Therefore,

$$M_{41} = 1 + (41 + 40 + \ldots + 1) = \frac{41 \times 42}{2} + 1 = 862.$$

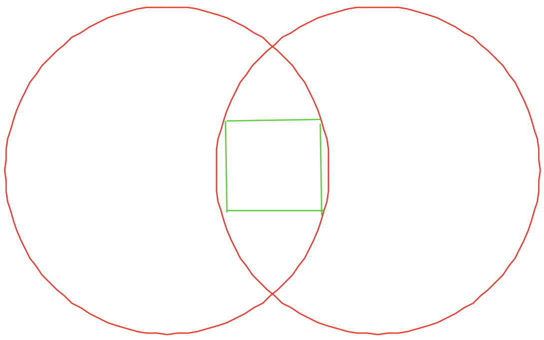
The minimum number of regions on the plane separated by k lines is when all the 41 lines are parallel, which give $m = 42$.

Therefore, $M + m = 904$.

Question S-19 [5 points]

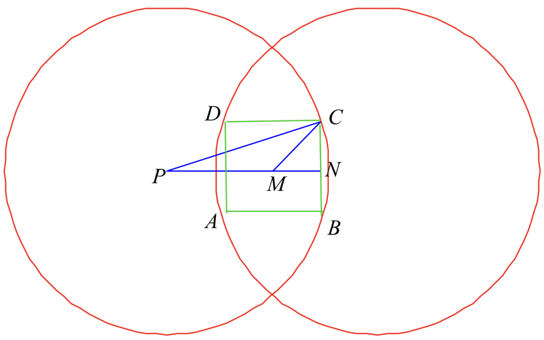
The figure below shows two circles with same radius $R = 85$. If the distance between the two centers is 142, find the area of the largest square that can be inscribed in the region where the two circles overlap.

下图中，两个圆的半径都是 $R = 85$，圆心的距离是 142。求内含在两圆重叠部分的最大正方形的面积。



Answer: [676]

Solutions:



$PC = 85$, $PM = \dfrac{142}{2} = 71$.
Let $MN = a$. Then $CN = a$.

$$a^2 + (a + 71)^2 = 85^2$$

$$2a^2 + 142a - 2184 = 0$$

$$a^2 + 71a - 1092 = 0$$

$$(a - 13)(a + 84) = 0$$

This shows that $a = 13$ and hence the area of the largest square is $4a^2 = 676$.

Question S-20 [5 points]

Given that k is a constant and the function

$$f(x)=\begin{cases}\dfrac{(x+1)^{29}-1}{x}, & x\neq 0\\ k, & x=0\end{cases}$$

is differentiable. Find $f'(0)$.

已知 k 是一常数且函数

$$f(x)=\begin{cases}\dfrac{(x+1)^{29}-1}{x}, & x\neq 0\\ k, & x=0\end{cases}$$

可微，求 $f'(0)$。

Answer: [406]

Solutions:

Since f is differentiable, it is continuous. Hence,

$$k=f(0)=\lim_{x\to 0}f(x)=\lim_{x\to 0}\frac{(x+1)^{29}-1}{x}=\lim_{x\to 0}29(x+1)^{28}=29.$$

$$\begin{aligned}f'(0)&=\lim_{x\to 0}\frac{f(x)-f(0)}{x}\\&=\lim_{x\to 0}\frac{\frac{(x+1)^{29}-1}{x}-29}{x}\\&=\lim_{x\to 0}\frac{(x+1)^{29}-1-29x}{x^2}\\&=\lim_{x\to 0}\frac{29(x+1)^{28}-29}{2x}\\&=\lim_{x\to 0}\frac{29\times 28(x+1)^{27}}{2}\\&=406\end{aligned}$$

Question S-21 [6 points]

Given that $a_1, a_2, \ldots, a_{1000}$ is an arithmetic sequence with integer terms, and the common difference is 1111. If the last three digits of a_{148} is 099, and the last three digits of a_n is 901, find n.

已知 a_1，a_2，$\ldots$，a_{1000} 是一公差为 1111 的等差数列，且此数列的每一项都是整数。若 a_{148} 的最后三位数是 099，a_n 的最后三位数是 901，求 n。

Answer: [930]

Solutions:

Let $a = a_1$ and $d = a_2 - a_1 = 1111$. Then

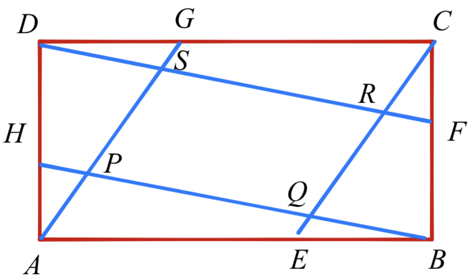
$$\begin{aligned}
a + 147d \equiv& 99 \mod 1000 \\
a + (n-1)d \equiv& 901 \mod 1000 \\
1111(n-148) \equiv& 802 \mod 1000 \\
111(n-148) \equiv& 802 \mod 1000 \\
999(n-148) \equiv& 7218 \mod 1000 \\
n - 148 \equiv& 1000 - 218 \mod 1000 \\
n \equiv& 930 \mod 1000
\end{aligned}$$

Since n has to be less than 1000, $n = 930$.

Question S-22 [6 points]

In the figure shown below, $ABCD$ is a rectangle, $AG//CE$, $BH//DF$, $AH : AD = 7 : 85$, $BE : BA = 3 : 11$. Given that the ratio of the area of parallelogram $PQRS$ to the area of rectangle $ABCD$ is $m : n$, where m and n are positive integers. Find the smallest possible value of n.

下图中，$ABCD$ 是长方形，$AG//CE$，$BH//DF$，$AH : AD = 7 : 85$，$BE : BA = 3 : 11$。已知平行四边形 $PQRS$ 的面积与长方形 $ABCD$ 的面积之比是 $m : n$，其中 m，n 是正整数，求 n 的最小可能值。



Answer: [239]

Solutions:

Let $k = \dfrac{7}{85}$, $l = \dfrac{3}{11}$, $AB = p$, $AD = q$.

Putting a coordinate system so that A is at the origin, B is on positive x-axis, and D is on positive y-axis. Then we have

$$A = (0,0), \quad B = (p,0), \quad C = (p,q), \quad D = (0,q), \quad G = (lp,q), \quad H = (0,kq).$$

The equations of AG and BH are respectively $y = \dfrac{q}{lp}x$, $y = -\dfrac{kq}{p}(x-p)$.

Solving these two equations give the coordinates of P as $(x_1, y_1) = \left(\dfrac{kl}{1+kl}p, \dfrac{k}{1+kl}q\right)$.

The equation of DF is $y - q = -\dfrac{kq}{p}x$. Solving this with $y = \dfrac{q}{lp}x$ give the coordinates of S as $(x_2, y_2) = \left(\dfrac{l}{1+kl}p, \dfrac{1}{1+kl}q\right)$.

By symmetry, the coordinates of Q is

$$(x_3, y_3) = \left(p - \frac{l}{1+kl}p, q - \frac{1}{1+kl}q\right) = \left(\frac{1+kl-l}{1+kl}p, \frac{kl}{1+kl}q\right).$$

The area of parallelogram $PQRS$ is twice the area of ΔPQS. Hence, the area of parallelogram

$PQRS$ is

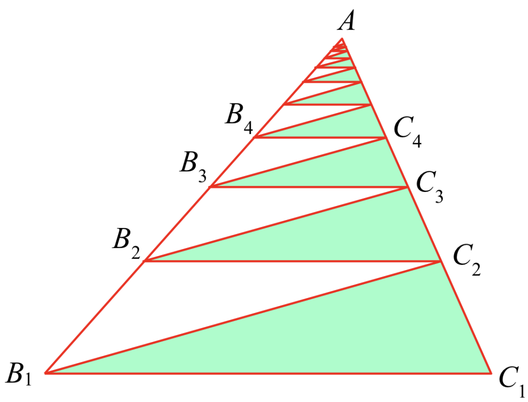
$$-\begin{vmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{vmatrix} = \frac{(1-k)(1-l)}{1+kl}pq = \frac{156}{239}pq.$$

This shows that $\frac{m}{n} = \frac{156}{239}$. Since 156 and 239 are relatively prime, the smallest possible value of n is 239.

Question S-23 [6 points]

In the figure shown below, $B_1, B_2, B_3, \ldots$ and $C_1, C_2, C_3, \ldots$ are two sequences of points. The sequence of lines B_1C_1, B_2C_2, B_3C_3, $\ldots$ are parallel, and the sequence of lines B_1C_2, B_2C_3, $B_3C_4, \ldots$ are parallel. Given that $AB_2 : B_2B_1 = 11 : 8$, S_n is the area of $\Delta B_nC_nC_{n+1}$, S is the sum of the infinite series $S_1 + S_2 + S_3 + \ldots$, and T is the area of ΔAB_1C_1. If $S : T = m : n$, where m and n are relatively prime positive integers, find the value of $m + n$.

下图中，B_1，B_2，B_3，$\ldots$ 与 C_1，C_2，C_3，$\ldots$ 是两个序列的点。B_1C_1，B_2C_2，B_3C_3，$\ldots$ 这一序列的直线互相平行，B_1C_2，B_2C_3，B_3C_4，$\ldots$ 这一序列的直线也互相平行。已知 $AB_2 : B_2B_1 = 11 : 8$，S_n 为 $\Delta B_nC_nC_{n+1}$ 的面积，S 为无穷级数 $S_1 + S_2 + S_3 + \ldots$ 的和，T 为ΔAB_1C_1 的面积，$S : T = m : n$，其中 m 与 n 为互质的正整数。求 $m + n$ 的值。



Answer: [49]

Solutions:
Let $k = \dfrac{AB_2}{AB_1} = \dfrac{11}{19}$.
By similarity, $S_{n+1} = k^2 S_n$.

Hence,

$$S = S_1 + S_2 + S_3 + \ldots = \frac{S_1}{1 - k^2}.$$

On the other hand

$$\frac{S_1}{T} = \frac{C_1C_2}{AC_1} = \frac{8}{19}.$$

Hence,

$$\frac{m}{n} = \frac{8}{19} \frac{1}{1 - \frac{121}{361}} = \frac{19}{30}.$$

This gives $m = 19$, $n = 30$ and $m + n = 49$.

Question S-24 [6 points]

The number of new students in Lizhi Secondary School is not more than 1000. Ms Lim, Ms Zhang and Ms Yu independently divide these students into several groups for the activities they organize. Ms Lim divides the students into 9 groups evenly. Ms Zhang divides the students into 11 groups, but four of the groups have one student more than the other seven groups. Ms Yu divides the students into 13 groups, but six of the groups have one student less than the other seven groups. If the school wants to divide the students into classes with same number of students, each class cannot have more than 40 students, at least how many classes should there be?

励志中学的新生在 1000 人以内。林老师，张老师及余老师分别将这些新生分组进行活动。林老师将学生平分为 9 组。张老师将学生分成 11 组，有其中四组比其他七组多一位学生。余老师将学生分成 13 组，有其中六组比其他七组少一位学生。如果学校要将这些学生分班，每班人数要在 40 人以内，每班人数要一样，至少应该分成几班？

Answer: [43]

Solutions:

If n is the total number of students, then $n = 9m = 11k + 4$ for some positive integers m and k.

We have

$$\begin{aligned} 2k + 4 &\equiv 0 \mod 9 \\ k + 2 &\equiv 0 \mod 9 \\ k &\equiv 7 \mod 9 \end{aligned}$$

Hence, $k = 9l + 7$ and $n = 99l + 81$. We also have

$$\begin{aligned} 99l + 81 &\equiv 7 \mod 13 \\ 8l &\equiv 4 \mod 13 \\ 2l &\equiv 1 \mod 13 \\ l &\equiv 7 \mod 13 \end{aligned}$$

Hence, $l = 13q + 7$ and hence $n = 1287q + 774$.

Since the school has no more than 1000 new students, it has 774 students.

Since $774 = 2 \times 3 \times 3 \times 43$, and the number of students in a class must be a factor of 774. The largest factor of 774 less than 40 is 18. Hence, there should be at least 43 classes.

Question S-25 [6 points]

Let $\omega = \cos\frac{2\pi}{101} + i\sin\frac{2\pi}{101}$. Find the value of

$$(\omega - 1)(\omega^2 - 1)\ldots(\omega^{100} - 1).$$

设 $\omega = \cos\frac{2\pi}{101} + i\sin\frac{2\pi}{101}$。求

$$(\omega - 1)(\omega^2 - 1)\ldots(\omega^{100} - 1)$$

的值。

Answer: [101]

Solutions:

Notice that for any integer k,

$$(\omega^k)^{101} = (\omega^{101})^k = 1.$$

Hence, 1, ω, ω^2, ..., ω^{100} are the distinct roots of $x^{101} - 1 = 0$. This shows that

$$(x - \omega)(x - \omega^2)\ldots(x - \omega^{100}) = \frac{x^{101} - 1}{x - 1} = x^{100} + x^{99} + \ldots + x + 1.$$

Put $x = 1$, we have

$$101 = (1 - \omega)(1 - \omega^2)\ldots(1 - \omega^{100}) = (\omega - 1)(\omega^2 - 1)\ldots(\omega^{100} - 1).$$

Question S-26 [8 points]

A strictly increasing sequence is constructed from the positive integers that are relatively prime to 2020. Find the 356th term.

将与 2020 互质的正整数按由小到大的顺序排列成一个数列，求第 356 项。

Answer: [897]

Solutions:

Since $2020 = 2^2 \times 5 \times 101$, a positive integer is relatively prime to 2020 if and only if it is relatively prime to 2, 5 and 101.

Among positive integers not larger than N, the numbers of those that are relatively prime to 2020 is

$$f(N) = N - \left\lfloor \frac{N}{2} \right\rfloor - \left\lfloor \frac{N}{5} \right\rfloor - \left\lfloor \frac{N}{101} \right\rfloor + \left\lfloor \frac{N}{10} \right\rfloor + \left\lfloor \frac{N}{202} \right\rfloor + \left\lfloor \frac{N}{505} \right\rfloor - \left\lfloor \frac{N}{1010} \right\rfloor$$

The 356th term in the sequence is the smallest N such that $f(N) = 356$.

We notice that $f(800) = 317$, $f(900) = 357$. We are close now.

We find that $f(899) = 357$, $f(898) = 356$, $f(897) = 356$ and $f(896) = 355$.

This shows that the 356th term in the sequence is 897.

Question S-27 [8 points]

Let $S = \{47x + 54y \,|\, x, y \text{ are nonnegative integers}\}$. Among the positive integers less than 10000, how many of them are not in the set S?

设 $S = \{47x + 54y \,|\, x, y$ 是非负的整数$\}$。在小于 10000 的正整数中，有多少个不在 S 中?

Answer: [1219]

Solutions:

Notice that if k is an integer, the integer solutions (x, y) to $47x + 54y = k$ can be written as

$$x = 23k - 54n, \quad y = 47n - 20k$$

for some integer n.

In order that x and y are nonnegative integers, we must have

$$\frac{54}{23}n \le k \le \frac{47}{20}n$$

for some nonnegative integers n.

Equivalently, there must be a nonnegative integer n so that

$$\frac{20}{47}k \le n \le \frac{23}{54}k.$$

Notice that if $k > 2538$,

$$\frac{23}{54}k - \frac{20}{47}k = \frac{k}{2538} > 1.$$

So there is always one integer n so that

$$\frac{20}{47}k \le n \le \frac{23}{54}k.$$

Hence, we only need to count how many positive integers not larger than 2538 that are in S. This is the same as counting for each $1 \le n \le 1080$, how many integers k are such that

$$\frac{54}{23}n \le k \le \frac{47}{20}n$$

Notice that when $n \le 1080$,

$$\frac{47}{20}(n-1) < \frac{54}{23}n,$$

and so no k would be double counted.

If $\lfloor x \rfloor$ is the largest integer not larger than x, and $\lceil x \rceil$ is the smallest integer not smaller than x, we find that the number of positive integers k not larger than 2538 that are in S is

$$\sum_{n=1}^{1080}\left(\left\lfloor \frac{47}{20}n \right\rfloor - \left\lceil \frac{54}{23}n \right\rceil + 1\right) = \sum_{n=1}^{1080}\left(\left\lfloor \frac{7}{20}n \right\rfloor - \left\lceil \frac{8}{23}n \right\rceil\right) + 1080$$

$$\begin{aligned}\sum_{n=1}^{1080}\left\lfloor \frac{7}{20}n \right\rfloor &= \sum_{k=1}^{54}\sum_{r=1}^{20}\left\lfloor \frac{7}{20}\left(20(k-1)+r\right)\right\rfloor \\ &= \sum_{k=1}^{54}\sum_{r=1}^{20}\left(7k-7+\left\lfloor \frac{7}{20}r \right\rfloor\right) \\ &= \sum_{k=1}^{54}(140k-76)\end{aligned}$$

$$\begin{aligned}\sum_{n=1}^{1080}\left\lceil \frac{8}{23}n \right\rceil &= \sum_{k=1}^{47}\sum_{r=1}^{23}\left\lceil \frac{8}{23}\left(23(k-1)+r\right)\right\rceil - 376 \\ &= \sum_{k=1}^{47}\sum_{r=1}^{23}\left(8k-8+\left\lceil \frac{8}{23}r \right\rceil\right) - 376 \\ &= \sum_{k=1}^{47}(184k-77) - 376\end{aligned}$$

The number of positive integers k not larger than 2538 that are in S is

$$140 \times \frac{54 \times 55}{2} - 76 \times 54 - 184 \times \frac{47 \times 48}{2} + 77 \times 47 + 376 + 1080 = 1319.$$

Therefore, the number of positive integers not in S is $2538 - 1319 = 1219$.

Question S-28 [8 points]

Given that n is a positive integer, and $N = 2040 + n$. If N has exactly 10 positive factors, find the smallest possible value of n.

已知 n 是一正整数，$N = 2040 + n$。若 N 恰好有 10 个正的因数，求 n 的最小可能值。

Answer: [56]

Solutions:

If the prime factorization of N is $N = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$, where p_1, p_2, ..., p_k are distinct prime numbers, and a_1, a_2, ..., a_k are positive integers, then N has $(a_1 + 1)(a_2 + 1) \ldots (a_k + 1)$ positive factors.

Given that N has exactly $10 = 2 \times 5$ positive factors, N must be of the form p^9 or $p_1 p_2^4$.

Integers of the form p^9 in ascending orders of p are $512, 19683, \ldots$

Integers of the form $p_1 p_2^4$ are

- $16p_1$, where p_1 is a prime not equal to 2. The smallest such number larger than 2040 is 2096 when $p_1 = 131$.
- $81p_1$, where p_1 is a prime not equal to 3. The smallest such number larger than 2040 is 2349 when $p_1 = 29$.
- $625p_1$, where p_1 is a prime not equal to 5. The smallest such number larger than 2040 is 4375 when $p_1 = 7$.
- $7^4 = 2401$, so for $p_2 \geq 7$, N would be larger than 2040.

The smallest positive n is 56.

Question S-29 [8 points]

Let $P = 1 \times 3 \times \ldots \times 9999$ be the product of the odd positive integers less than 10000. Find the largest integer k such that 15^k divides P.

设 $P = 1 \times 3 \times \ldots \times 9999$ 是小于 10000 的正奇数的乘积。求最大的整数 k 使得 15^k 可以整除 P。

Answer: [1250]

Solutions:

P is divisible by 15^k if and only if it is divisible by 3^k and 5^k.

The multiples of 5 among all odd integers less than 10000 are

$$5, 15, 25, \ldots, 9995$$

There are 1000 of them.

The multiples of 25 among all odd integers less than 10000 are

$$25, 75, 125, \ldots, 9975$$

There are 200 of them.

The multiples of 125 among all odd integers less than 10000 are

$$125, 375, 625, \ldots, 9875$$

There are 40 of them.

The multiples of 625 among all odd integers less than 10000 are

$$625, 1875, 3125, \ldots, 9375$$

There are 8 of them.

The multiples of 3125 among all odd integers less than 10000 are

$$3125, 9375$$

There are 2 of them.

The largest k such that 5^k divides P is

$$1000 + 200 + 40 + 8 + 2 = 1250.$$

3^{1250} also divides P.

Hence, the largest k such that 15^k divides P is 1250.

Question S-30 [8 points]

Let $S = \{2, 5, 8, \ldots, 83\}$ be the set containing all positive integers less than 84 which leaves a remainder of 2 when divided by 3. For a set A that is a subset of S, define $f(A)$ to be the sum of the elements in A. If $f(A) = 83$, we say that A is awkward. Among all the subsets of S, how many of them are awkward?

设 $S = \{2, 5, 8, \ldots, 83\}$ 为小于 84，且除以 3 余 2 的正整数集合。对于 S 的子集合 A，定义 $f(A)$ 为 A 中所有元素的和。若 $f(A) = 83$，A 就被称为尴尬的集合。S 的子集中，有几个是尴尬的?

Answer: [97]

Solutions:

Let A be an awkward subset of S. Notice that if A has m elements, then

$$f(A) \geq 2 + 5 + \ldots + 3m - 1 = \frac{m(3m+1)}{2}.$$

For $f(A) = 83$, $m \leq 7$.

If the m elements of A are $3k_1 - 1$, $3k_2 - 1$, ..., $3k_m - 1$ with $1 \leq k_1 < k_2 < \ldots < k_m \leq 28$, then

$$3(k_1 + k_2 + \ldots + k_m) = 83 + m.$$

This shows that m should leave a remainder of 1 when divided by 3.

Hence, m can only have 1, 4 or 7 elements.

If $m = 1$, then $A = \{83\}$. There is only one such A.

If $m = 4$,

$$k_1 + k_2 + k_3 + k_4 = 29.$$

Let

$$\begin{aligned}
k_1 &= a + 1, \\
k_2 &= a + 1 + b + 1, \\
k_3 &= a + 1 + b + 1 + c + 1, \\
k_4 &= a + 1 + b + 1 + c + 1 + d + 1
\end{aligned}$$

The solutions for (k_1, k_2, k_3, k_4) are in one-to-one correspondence with the nonnegative integer solutions to

$$4a + 3b + 2c + d = 19.$$

The number of solutions is

$$\begin{aligned}\sum_{a=0}^{4}\sum_{b=0}^{\left\lfloor\frac{19-4a}{3}\right\rfloor}\sum_{c=0}^{\left\lfloor\frac{19-4a-3b}{2}\right\rfloor}1=&\sum_{a=0}^{4}\sum_{b=0}^{\left\lfloor\frac{19-4a}{3}\right\rfloor}\left(\left\lfloor\frac{19-4a-3b}{2}\right\rfloor+1\right)\\ =&\sum_{b=0}^{\left\lfloor\frac{19}{3}\right\rfloor}\left(\left\lfloor\frac{19-3b}{2}\right\rfloor+1\right)+\sum_{b=0}^{\left\lfloor\frac{15}{3}\right\rfloor}\left(\left\lfloor\frac{15-3b}{2}\right\rfloor+1\right)+\sum_{b=0}^{\left\lfloor\frac{11}{3}\right\rfloor}\left(\left\lfloor\frac{11-3b}{2}\right\rfloor+1\right)\\ &+\sum_{b=0}^{\left\lfloor\frac{7}{3}\right\rfloor}\left(\left\lfloor\frac{7-3b}{2}\right\rfloor+1\right)+\sum_{b=0}^{\left\lfloor\frac{3}{3}\right\rfloor}\left(\left\lfloor\frac{3-3b}{2}\right\rfloor+1\right)\\ =&94\end{aligned}$$

If $m=7$,

$$k_1+k_2+k_3+k_4+k_5+k_6+k_7=30.$$

Let

$$\begin{aligned}k_1=&a+1,\\ k_2=&a+1+b+1,\\ k_3=&a+1+b+1+c+1,\\ k_4=&a+1+b+1+c+1+d+1,\\ k_5=&a+1+b+1+c+1+d+1+e+1,\\ k_6=&a+1+b+1+c+1+d+1+e+1+f+1,\\ k_7=&a+1+b+1+c+1+d+1+e+1+f+1+g+1\end{aligned}$$

The solutions for $(k_1,k_2,k_3,k_4,k_5,k_6,k_7)$ are in one-to-one correspondence with the nonnegative integer solutions to

$$7a+6b+5c+4d+3e+2f+g=2.$$

Then we must have $a=b=c=d=e=0$, and $2f+g=2$. Then we can only have $(f,g)=(0,2)$ or $(1,0)$. Two cases.
Altogether, there are 97 awkward sets.