Question S-01 [5 points]

Given that the function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(3x + 2) = 2x + 2021$, find $f(2021)$.

已知函数 $f : \mathbb{R} \to \mathbb{R}$ 满足 $f(3x + 2) = 2x + 2021$，求 $f(2021)$。

Answer: [3367]

Solutions:

$$
\begin{aligned}
3x + 2 &= 2021 \\
x &= 673 \\
2x + 2021 &= 3367
\end{aligned}
$$

Question S-02 [5 points]

In a game, each of the eight participants is given a card to write her own name on it. They then give the cards to the host. After mixing up the cards, the host randomly distributes the cards to the participants, each participant gets one. If x is the probability that exactly one of the participants does not get back the card with her own name written on it, find the value of $8!\,x$.

在一个游戏中，八位参与者各在一张卡片上写上自己的名字。她们之后将卡片交给主持人。主持人弄乱了卡片后，再任意将卡片发给这八位参与者，每人一张。若 x 为恰有一位参与者没有拿到写上自己名字的卡片的概率，求 $8!\,x$ 的值。

Answer: [0]

Solutions:

If exactly one participant does not get back the card with her own name written on it, then all the other seven participants get back their own cards. This cannot happen. Hence, the probability is 0.

Question S-03 [5 points]

Given that a, b, c are three positive numbers with $(a+b-c):(b+c-a):(c+a-b)=6:7:9$ and $a+b+c=176$. Find b .

已知a, b, c 三个正数满足 $(a+b-c):(b+c-a):(c+a-b)=6:7:9$，且 $a+b+c=176$。求b。

Answer: [52]

Solutions:

$$
\begin{aligned}
a+b-c &= 6k \\
b+c-a &= 7k \\
c+a-b &= 9k \\
a+b+c &= 22k = 176 \\
k &= 8 \\
2b &= 13k = 13 \times 8 \\
b &= 52
\end{aligned}
$$

Question S-04 [5 points]

Let x be the number of ways to permute the letters in the word INSPECTION such that the two N's are not adjacent, find $\frac{x}{720}$.

设 x 为将INSPECTION 一字中的英文字母排列，两个N 不相邻的排列数。求 $\frac{x}{720}$。

Answer: [1008]

Solutions:

The number of permutations of the word INSPECTION is $\frac{10!}{2!\,2!}$.

The number of permutations of the word INSPECTION where the two N's are adjacent is $\frac{9!}{2!}$.

The number of permutations of the word INSPECTION where the two N's are not adjacent is

$$x = \frac{10!}{2!2!} - \frac{9!}{2!} = 9!\left(\frac{10}{4} - \frac{1}{2}\right) = 6! \times 7 \times 8 \times 9 \times 2$$

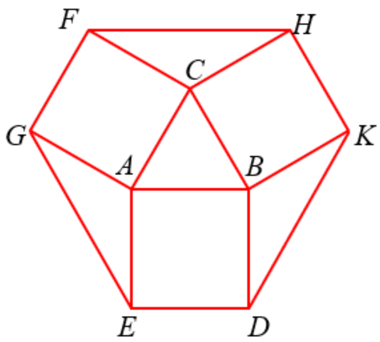
Hence,

$$\frac{x}{720} = 1008.$$

Question S-05 [5 points]

In the figure below, ABC is an equilateral triangle. $ABDE$, $ACFG$ and $BCHK$ are squares. Given that $AB = 2$ and x is the area of the hexagon $DEGFHK$. Find the largest integer less than or equal to x.

下图中，ABC 是一个等边三角形。$ABDE$，$ACFG$ 及 $BCHK$ 是正方形。已知 $AB = 2$ 且 x 是六边形 $DEGFHK$ 的面积。求小于或等于 x 的最大整数。



Answer: [18]

Solutions:

$$\angle FCH = 180° - 60° = 120°$$

$$\angle CHF = \angle CFH = 30°$$

$$CH = BC = AB = 2$$

$$FH = 2\sqrt{3}$$

$$\text{Area of } \Delta ABC = \frac{1}{2} \times 2^2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$$

$$\text{Area of } \Delta CFH = \frac{1}{2} \times 2^2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$$

$$x = 4\sqrt{3} + 3 \times 4 = 12 + 4\sqrt{3}$$

$$\lfloor x \rfloor = 18$$

Question S-06 [5 points]

Given that the set R consists of all pairs of real numbers (x, y) satisfying

$$\begin{aligned} 2x + y - 2 &\geq 0 \\ 2x - 3y + 6 &\geq 0 \\ x - 2y - 1 &\leq 0 \\ x + y - 7 &\leq 0 \end{aligned}$$

Find the maximum value of $x + 3y$ on R.

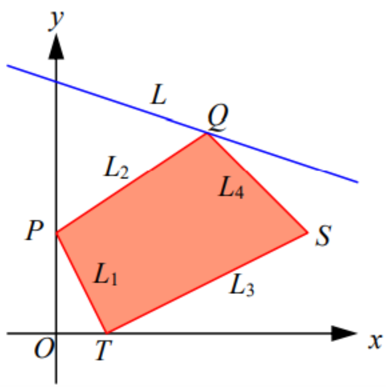
已知集合 R 的元素是所有满足不等式组

$$\begin{aligned} 2x + y - 2 &\geq 0 \\ 2x - 3y + 6 &\geq 0 \\ x - 2y - 1 &\leq 0 \\ x + y - 7 &\leq 0 \end{aligned}$$

的实数对 (x, y)。求 $x + 3y$ 在 R 上的最大值。

Answer: [15]

Solutions:



Let $P(0, 2)$, $Q(3, 4)$, $S(5, 2)$ and $T(1, 0)$. R is the polygonal region $PQST$ bounded by the lines

$$\begin{aligned} L_1 &: \quad 2x + y - 2 = 0 \\ L_2 &: \quad 2x - 3y + 6 = 0 \\ L_3 &: \quad x - 2y - 1 = 0 \\ L_4 &: \quad x + y - 7 = 0 \end{aligned}$$

We find that the maximum of $x + 3y$ on R appears at the point $Q(3, 4)$, and the value is 15.

Question S-07 [5 points]

Given that $\log_{10} 2 = 0.3010$, how many digits does the number 2^{2021} have?

已知 $\log_{10} 2 = 0.3010$， 2^{2021} 这个数有几位数字?

Answer: [609]

Solutions:

$$\log_2 2^{2021} = 2021 \log_{10} 2 = 608.321$$

Hence,

$$10^{608} < 2^{2021} < 10^{609}.$$

Therefore, 2^{2021} has 609 digits.

Question S-08 [5 points]

Find the maximum value of the function $f(x) = \dfrac{5x^2+10x+71}{x^2+2x+7}$.

求函数 $f(x) = \dfrac{5x^2+10x+71}{x^2+2x+7}$ 的最大值。

Answer: [11]

Solutions:

$$\begin{aligned} f(x) =& \frac{5(x^2+2x+7)+36}{x^2+2x+7} \\ =& 5 + \frac{36}{(x+1)^2+6} \end{aligned}$$

When $x = -1$, $f(x)$ has maximum value 11.

Question S-09 [5 points]

Four people, P, Q, R and S, are accused in a trial. It is known that

- If P is guilty, then Q is guilty.
- If Q is guilty, then either R is guilty or P is not guilty.
- If S is guilty, then P is guilty and R is not guilty.
- If S is not guilty, then P is guilty.

Among these four accused P, Q, R, S, how many of them are guilty?

在一次审讯中有四位被告P，Q，R及S。已知

- 如果P是有罪的，则Q也是有罪的。
- 如果Q是有罪的，则R是有罪的或P是无罪的。
- 如果S是有罪的，则P是有罪的且R是无罪的。
- 若果S是无罪的，则P是有罪的。

P，Q，R，S四位被告中，有多少位有罪?

Answer: [3]

Solutions:

We analyse the two cases where P is guilty and P is not guilty.

If P is guilty, then Q is guilty. Then R is guilty or P is not guilty. But we already know that P is guilty. Hence, R must be guilty.

Now if S is guilty, R is not guilty. But we already infer that R is guilty. Hence S is not guilty. This implies that P, Q, R are guilty but S is not guilty.

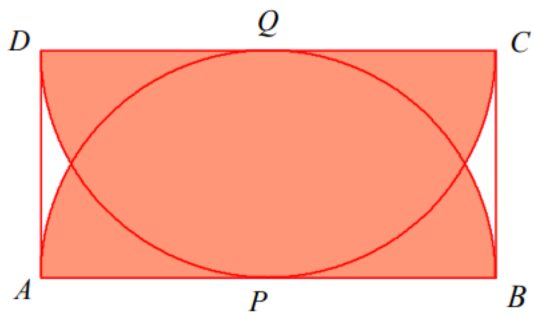
If P is not guilty, then S is guilty. (Since if S is not guilty, P must be guilty). But then P must be guilty. This is a contradiction.

Hence, P, Q, R are guilty and S is not guilty.

Question S-10 [5 points]

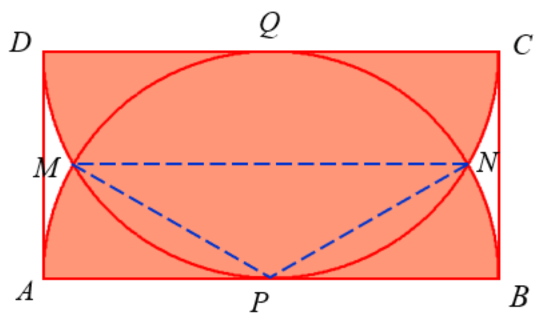
In the figure below, $ABCD$ is a rectangle with $AB = 12$, $AD = 6$. The arcs AQB and CPD are semicircles with diameters AB and CD respectively. If the area of the shaded region is x, find the integer that is closest to x.

下图中，$ABCD$ 是一个长方形，$AB = 12$，$AD = 6$。弧 AQB 及弧 CPD 为半圆，直径分别为 AB 及 CD。若阴影部分的面积为 x，求最靠近 x 的整数。



Answer: [69]

Solutions:



The area of the shaded region is equal to the sum of the areas enclosed by the two semicircles, minus the area of their common region.

Notice that $\angle MPN = 120°$.

Area of the common region is

$$=2\,(\text{area of sector } MPNQ \;-\; \text{area of } \Delta MPN)$$
$$=2\left(\frac{1}{3}\times\pi\times 6^2-\frac{1}{2}\times 6\sqrt{3}\times 3\right)$$
$$=24\pi-18\sqrt{3}$$

Hence, the area of the shaded region is

$$x = 36\pi - \left(24\pi - 18\sqrt{3}\right) = 12\pi + 18\sqrt{3} = 68.88.$$

The integer that is closest to x is 69.

Question S-11 [5 points]

Given that $\log_a b = 317$, find the value of $\log_{\sqrt{a}}\left(\sqrt{a^5b^3}\right)$.

已知 $\log_a b = 317$，求 $\log_{\sqrt{a}}\left(\sqrt{a^5b^3}\right)$ 的值。

Answer: [956]

Solutions:

$$\log_{\sqrt{a}}\left(\sqrt{a^5b^3}\right) = \frac{\frac{1}{2}\left(\log_a a^5 + \log_a b^3\right)}{\log_a \sqrt{a}} = \frac{\frac{5}{2} + \frac{3}{2} \times 317}{\frac{1}{2}} = 956$$

Question S-12 [5 points]

Find the sum of the integers n that satisfy the inequality $12n^2 < 383n - 2020$.

求满足不等式 $12n^2 < 383n - 2020$ 的整数 n 之和。

Answer: [304]

Solutions:

$$12n^2 - 383n + 2020 < 0$$

$$(3n - 20)(4n - 101) < 0$$

$$\frac{20}{3} < n < \frac{101}{4}$$

The sum of the integers n that satisfy the inequality is

$$7 + 8 + \ldots + 24 + 25 = \frac{19}{2}(7 + 25) = 304.$$

Question S-13 [5 points]

If a and b are two distinct positive numbers such that $11a - \frac{7}{b} = 11b - \frac{7}{a}$, find the value of $1001ab$.

若 a 及 b 是相异的正数且 $11a - \frac{7}{b} = 11b - \frac{7}{a}$，求 $1001ab$ 的值。

Answer: [637]

Solutions:

$$11(a-b) = 7\left(\frac{1}{b} - \frac{1}{a}\right) = \frac{7(a-b)}{ab}$$

Since $a - b \neq 0$, we have

$$ab = \frac{7}{11}$$

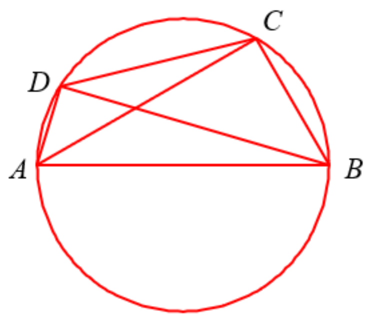
Hence,

$$1001ab = 637$$

Question S-14 [5 points]

In the figure shown below, $ABCD$ is a cyclic quadrilateral, AB is a diameter of the circle. Given that $AD = 12$, $BC = 19$, $AC = x$, $BD = y$, find the value of $y^2 - x^2$.

下图中，$ABCD$ 是一圆内接四边形，AB 是圆的直径。已知 $AD = 12$，$BC = 19$，$AC = x$，$BD = y$，求 $y^2 - x^2$ 的值。



Answer: [217]

Solutions:

Let $AB = d$. Since $\angle ADB = \angle ACB = 90°$, we have

$$d^2 = x^2 + 19^2 = y^2 + 12^2.$$

Therefore,

$$y^2 - x^2 = 217.$$

Question S-15 [5 points]

If n is a positive integer such that

$$2021\left[\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\ldots+\frac{1}{(2n-1)(2n+1)}\right]$$

is also an integer, find the largest possible value of n.

若 n 是一正整数使得

$$2021\left[\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\ldots+\frac{1}{(2n-1)(2n+1)}\right]$$

也是整数，求 n 的最大可能值。

Answer: [1010]

Solutions:

$$\begin{aligned}&2021\left[\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\ldots+\frac{1}{(2n-1)(2n+1)}\right]\\=&\frac{2021}{2}\left[1-\frac{1}{3}+\frac{1}{3}-\frac{1}{5}+\frac{1}{5}-\frac{1}{7}+\ldots+\frac{1}{2n-1}-\frac{1}{2n+1}\right]\\=&\frac{2021}{2}\left[1-\frac{1}{2n+1}\right]\\=&\frac{2021n}{2n+1}\end{aligned}$$

Since n is relatively prime to $2n+1$, $2n+1$ must divide 2021.

The largest odd factor of 2021 is 2021.

Hence, the largest possible value of n is 1010.

Question S-16 [5 points]

If k is an integer such that $(k-8)x^2-(2k-5)x+(k-9)$ is negative for all real numbers x, find the largest possible value of k.

若 k 是一整数且对于所有的实数 x，$(k-8)x^2-(2k-5)x+(k-9)$ 的值都是负的，求 k 的最大可能值。

Answer: [5]

Solutions:

$$(2k-5)^2-4(k-8)(k-9)<0$$

$$48k<263$$

$$k<\frac{263}{48}$$

The largest possible value of k is 5.

Question S-17 [5 points]

Let a, b, c be the three roots of the equation $x^3 - 99x - 101 = 0$. Find the value of $a^3 + b^3 + c^3$.

设 a，b，c 为方程式 $x^3 - 99x - 101 = 0$ 的三个根。求 $a^3 + b^3 + c^3$ 的值。

Answer: [303]

Solutions:

$$a^3 = 99a + 101$$
$$b^3 = 99b + 101$$
$$c^3 = 99c + 101$$
$$a^3 + b^3 + c^3 = 99(a + b + c) + 303 = 303$$

Question S-18 [5 points]

Under a rotation with respect to the point $P(a,b)$, the image of the point $A(11,5)$ is $A'(9,11)$. Find the value of $3b-a$.

已知经过以点 $P(a,b)$ 为中心的旋转后，点 $A(11,5)$ 的像是 $A'(9,11)$，求 $3b-a$ 的值。

Answer: [14]

Solutions:

$$AP = A'P$$

$$(a-11)^2+(b-5)^2=(a-9)^2+(b-11)^2$$

$$a^2+b^2-22a-10b+146=a^2+b^2-18a-22b+202$$

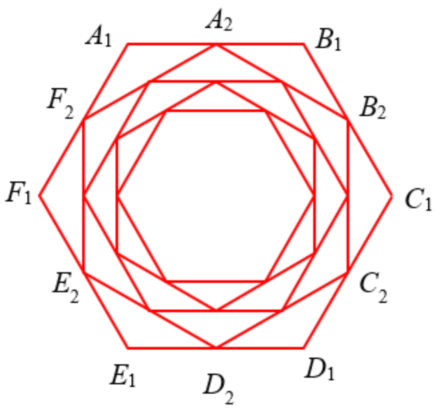
$$12b-4a=56$$

$$3b-a=14$$

Question S-19 [5 points]

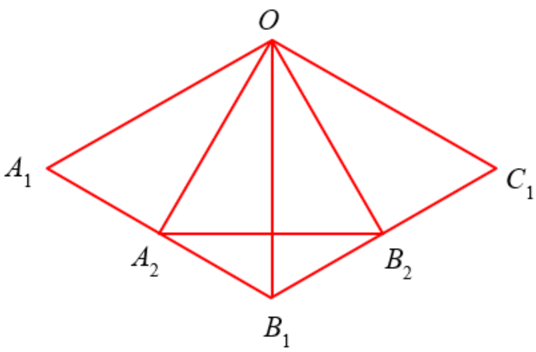
In the figure below, $A_1B_1C_1D_1E_1F_1$ is a regular hexagon. The midpoints of the sides of $A_1B_1C_1D_1E_1F_1$ are joined to form the hexagon $A_2B_2C_2D_2E_2F_2$. The midpoints of the sides of $A_2B_2C_2D_2E_2F_2$ are joined to form the hexagon $A_3B_3C_3D_3E_3F_3$. This process continues to form the hexagons $A_nB_nC_nD_nE_nF_n$ for all $n \geq 2$. If the area of the hexagon $A_nB_nC_nD_nE_nF_n$ is S_n, and $S_1 = 228$, find the sum of the infinite series $S_1 + S_2 + S_3 + \ldots$.

如下图所示，$A_1B_1C_1D_1E_1F_1$ 是一个正六边形。将 $A_1B_1C_1D_1E_1F_1$ 各边的中点连起来就得到六边形 $A_2B_2C_2D_2E_2F_2$。将 $A_2B_2C_2D_2E_2F_2$ 各边的中点连起来就得到六边形 $A_3B_3C_3D_3E_3F_3$。不断重复这过程以得到所有 $n \geq 2$ 时的六边形 $A_nB_nC_nD_nE_nF_n$。若六边形 $A_nB_nC_nD_nE_nF_n$ 的面积是 S_n，且 $S_1 = 228$，求无穷级数 $S_1 + S_2 + S_3 + \ldots$ 的和。



Answer: [912]

Solutions:



Let O be the center of the hexagon. Then OA_1B_1, OA_2B_2, OB_1C_1 are regular triangles, as shown in the figure above.

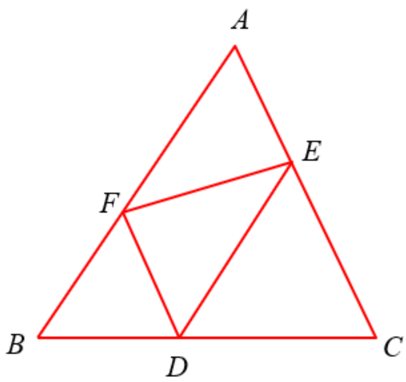
Hence, OA_2 is perpendicular to A_1B_1.
This implies that $\dfrac{OA_2}{OA_1} = \dfrac{\sqrt{3}}{2}$, and therefore, $\dfrac{S_2}{S_1} = \dfrac{3}{4}$.
Similarly, $\dfrac{S_{n+1}}{S_n} = \dfrac{3}{4}$.
Hence,

$$S_1 + S_2 + S_3 + \ldots = \frac{228}{1 - \frac{3}{4}} = 912.$$

Question S-20 [5 points]

In the figure shown below, $AF : FB = 4 : 3$, $AE : EC = 3 : 5$, $BD : DC = 4 : 5$. Given that the areas of ΔABC and ΔDEF are S_1 and S_2 respectively. If $S_1 : S_2 = m : n$, where m and n are relatively prime positive integers, find the value of $m + n$.

下图中，$AF : FB = 4 : 3$，$AE : EC = 3 : 5$，$BD : DC = 4 : 5$。已知ΔABC及ΔDEF 的面积分别为S_1及S_2。若m与n是互质的正整数使得$S_1 : S_2 = m : n$，求$m + n$ 的值。



Answer: [629]

Solutions:

$$\frac{AF}{AB} = \frac{4}{7}, \frac{AE}{AC} = \frac{3}{8}$$

$$\frac{S_{\Delta AEF}}{S_{\Delta ABC}} = \frac{\frac{1}{2} \times AE \times AF \times \sin A}{\frac{1}{2} \times AB \times AC \times \sin A} = \frac{12}{56}$$

$$\frac{BF}{BA} = \frac{3}{7}, \frac{BD}{BC} = \frac{4}{9}$$

$$\frac{S_{\Delta BDF}}{S_{\Delta ABC}} = \frac{12}{63}$$

$$\frac{CD}{CB} = \frac{5}{9}, \frac{CE}{CA} = \frac{5}{8}$$

$$\frac{S_{\Delta CDE}}{S_{\Delta ABC}} = \frac{25}{72}$$

$$\frac{S_{\Delta DEF}}{S_{\Delta ABC}} = 1 - \frac{12}{56} - \frac{12}{63} - \frac{25}{72} = \frac{125}{504}$$

Hence, $m = 504$, $n = 125$, $m + n = 629$.

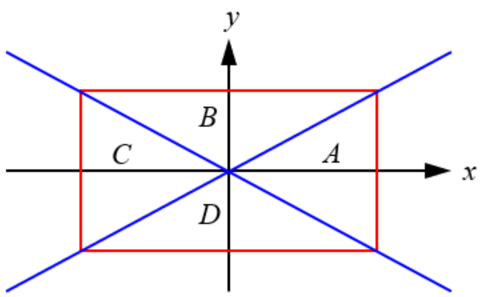
Question S-21 [6 points]

Find the area of the region in the plane defined by the inequality $|x+2y|+|x-2y| \leq 34$.

求平面上由不等式 $|x+2y|+|x-2y| \leq 34$ 所定义的区域的面积。

Answer: [578]

Solutions:



The lines $x = 2y$ and $x = -2y$ divide the plane into 4 regions.

Region A:

$x > 2y$ and $x > -2y$

$$|x+2y|+|x-2y| \leq 34$$
$$x+2y+x-2y \leq 34$$
$$x \leq 17$$

Region B:

$x < 2y$ and $x < -2y$

$$|x+2y|+|x-2y| \leq 34$$
$$x+2y-x+2y \leq 34$$
$$y \leq \frac{17}{2}$$

Region C:

$x < 2y$ and $x < -2y$

$$|x+2y|+|x-2y| \leq 34$$
$$-x-2y-x+2y \leq 34$$
$$x \geq -17$$

Region D:

$x > 2y$ and $x < -2y$

$$
\begin{aligned}
|x+2y| + |x-2y| &\leq 34 \\
-x - 2y + x - 2y &\leq 34 \\
y &\geq -\frac{17}{2}
\end{aligned}
$$

Hence, the region defined by is a rectangle of length 34 and width 17, and the area is 578.

Question S-22 [6 points]

If $S=\sum_{k=1}^{112}\frac{(-1)^{k+1}}{\sqrt{4k^2-1}\left(\sqrt{2k+1}-\sqrt{2k-1}\right)}$, find the value of $120S$.

若 $S=\sum_{k=1}^{112}\frac{(-1)^{k+1}}{\sqrt{4k^2-1}\left(\sqrt{2k+1}-\sqrt{2k-1}\right)}$，求 $120S$ 的值。

Answer: [56]

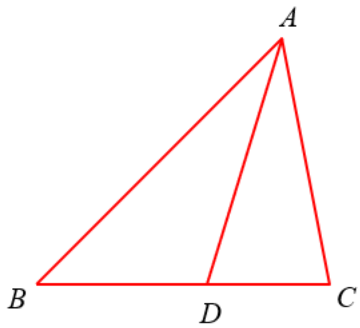
Solutions:

$$
\begin{aligned}
S &=\sum_{k=1}^{112}\frac{(-1)^{k+1}}{\sqrt{4k^2-1}\left(\sqrt{2k+1}-\sqrt{2k-1}\right)}\\
&=\frac{1}{2}\sum_{k=1}^{112}\frac{(-1)^{k+1}\left(\sqrt{2k+1}+\sqrt{2k-1}\right)}{\sqrt{(2k+1)(2k-1)}}\\
&=\frac{1}{2}\sum_{k=1}^{112}(-1)^{k+1}\left(\frac{1}{\sqrt{2k-1}}+\frac{1}{\sqrt{2k+1}}\right)\\
&=\frac{1}{2}\left[\left(1+\frac{1}{\sqrt{3}}\right)-\left(\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{5}}\right)+\ldots+\left(\frac{1}{\sqrt{221}}+\frac{1}{\sqrt{223}}\right)-\left(\frac{1}{\sqrt{223}}+\frac{1}{\sqrt{225}}\right)\right]\\
&=\frac{1}{2}\left(1-\frac{1}{15}\right)\\
&=\frac{7}{15}\\
& 120S=56
\end{aligned}
$$

Question S-23 [6 points]

In the figure below, AD bisects $\angle BAC$. Given that $AB = 9$, $BC = 10$, $AC = 6$. If $AD = x$, find the value of x^2.

下图中，AD 平分 $\angle BAC$。已知 $AB = 9$，$BC = 10$，$AC = 6$。若 $AD = x$，求 x^2 的值。



Answer: [30]

Solutions:

$$\frac{BD}{CD} = \frac{AB}{AC} = \frac{3}{2}$$
$$BD = 6,\ CD = 4$$

Let $\angle ADC = \theta$. Then

$$6^2 = 4^2 + x^2 - 8x\cos\theta$$
$$9^2 = 6^2 + x^2 + 12x\cos\theta$$
$$3 \times 36 + 2 \times 81 = 3 \times 16 + 3x^2 + 2 \times 36 + 2x^2$$
$$x^2 = 30$$

Question S-24 [6 points]

Find the largest integer less than 1000 which has exactly 7 positive factors.

求小于 1000 且恰有 7 个正因子的最大整数。

Answer: [729]

Solutions:

If N is an integer which has exactly 7 positive factors, then $N = p^6$ for some prime p. The largest such integer less than 1000 is $3^6 = 729$.

Question S-25 [6 points]

Given that $a_0 = 1$, $a_1 = \frac{1}{2}$, and for all $n \geq 1$, $a_{n+1} = \frac{a_{n-1}}{1 + na_{n-1}a_n}$. Find $\frac{1}{a_{99}a_{100}}$.

已知 $a_0 = 1$，$a_1 = \frac{1}{2}$，且对于所有的 $n \geq 1$，$a_{n+1} = \frac{a_{n-1}}{1 + na_{n-1}a_n}$，求 $\frac{1}{a_{99}a_{100}}$。

Answer: [4952]

Solutions:

$$\frac{1}{a_{n+1}} = \frac{1}{a_{n-1}} + na_n$$

$$\frac{1}{a_{n+1}a_n} = \frac{1}{a_n a_{n-1}} + n$$

$$\begin{aligned}
\frac{1}{a_{100}a_{99}} - \frac{1}{a_1 a_0} &= \sum_{n=1}^{99} \left(\frac{1}{a_{n+1}a_n} - \frac{1}{a_n a_{n-1}} \right) \\
&= \sum_{n=1}^{99} n \\
&= \frac{99 \times 100}{2} \\
&= 4950
\end{aligned}$$

$$\frac{1}{a_{99}a_{100}} = 4952$$

Question S-26 [8 points]

Consider the 2021 fractions

$$\frac{1}{2021}, \frac{2}{2021}, \frac{3}{2021}, \ldots, \frac{2021}{2021}.$$

Let S be the set of these fractions which cannot be reduced to fractions with smaller denominators. For example, $\frac{3}{2021}$ is in S but $\frac{43}{2021}$ is not in S since $\frac{43}{2021} = \frac{1}{47}$. Find the sum of the elements in S.

考虑以下 2021 个分数：

$$\frac{1}{2021}, \frac{2}{2021}, \frac{3}{2021}, \ldots, \frac{2021}{2021}.$$

设集合 S 是这些分数中不能被约简成分母比较小的分数所组成的集合。例如，$\frac{3}{2021}$ 在 S 中，而 $\frac{43}{2021}$ 不在 S 中，因为 $\frac{43}{2021} = \frac{1}{47}$。求 S 中的元素之和。

Answer: [966]

Solutions:

Since $2021 = 43 \times 47$, $\frac{k}{2021}$ is in S if and only if $1 \le k \le 2021$ and k is relatively prime to 43 and 47.

The sum of the integers from 1 to 2021 is $\frac{2021 \times 2022}{2}$.

The sum of those divisible by 43 is

$$43\,(1 + 2 + \ldots + 47) = \frac{43 \times 47 \times 48}{2} = \frac{2021 \times 48}{2}$$

The sum of those divisible by 47 is

$$47\,(1 + 2 + \ldots + 43) = \frac{47 \times 43 \times 44}{2} = \frac{2021 \times 44}{2}$$

The integer 2021 is counted in both.

Hence, the sum of the elements in S is

$$\frac{1}{2021}\left(\frac{2021 \times 2022}{2} - \frac{2021 \times 48}{2} - \frac{2021 \times 44}{2} + 2021\right)$$

$$=1011 - 24 - 22 + 1$$

$$=966$$

Question S-27 [8 points]

How many pairs of positive integers (m, n) satisfy the equation $\frac{1}{m}+\frac{1}{n}=\frac{3}{10000}$?

有多少对正整数 (m, n) 满足方程式 $\frac{1}{m}+\frac{1}{n}=\frac{3}{10000}$?

Answer: [40]

Solutions:

$$\frac{1}{m}+\frac{1}{n}=\frac{3}{10000}$$

$$10000(m+n)=3mn$$

$$(3m-10000)(3n-10000)=2^8\times 5^8$$

Notice that $(3m-10000)$ and $(3n-10000)$ are integers not smaller than -9999. In order that their product is $2^8\times 5^8$, they must be both positive integers. We also observe that they are both congruent to 2 modulo 3.

Conversely, given two positive integers x and y that are congruent to 2 modulo 3 such that $xy=2^8\times 5^8$, we can solve $x=3m-10000$ and $y=3n-10000$ for positive integers m and n satisfying

$$\frac{1}{m}+\frac{1}{n}=\frac{3}{10000}$$

Since 2 and 5 are both congruent to 2 modulo 3, we find that $3m-10000$ must be of the form $2^a\times 5^b$, where $0\le a\le 8$, $0\le b\le 8$, and $a+b$ must be odd.

If $a=0,2,4,6,8$, b can be 1, 3, 5, 7.

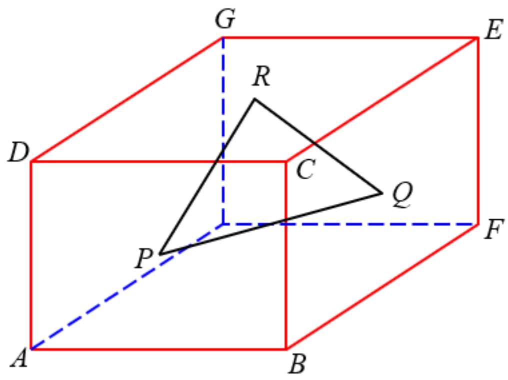
If $a=1,3,5,7$, b can be $0,2,4,6,8$.

These give $20+20=40$ pairs of positive integer solutions (m, n).

Question S-28 [8 points]

The figure below shows a rectangular box. Given that the points P, Q, R are the centers of the faces $ABCD$, $BCEF$ and $CDGE$ respectively. If the lengths of PQ, QR and PR are 5, $\sqrt{52}$ and $\sqrt{45}$ respectively, find the volume of the rectangular box.

下图所示是一个长方体。已知 P，Q，R 分别是平面 $ABCD$，$BCEF$ 及 $CDGE$ 的中心。若 PQ，QR 及 PR 的长分别为5，$\sqrt{52}$ 及 $\sqrt{45}$，求长方体的体积。



Answer: [576]

Solutions:

Let $AB = 2x$, $BF = 2y$, $AD = 2z$.

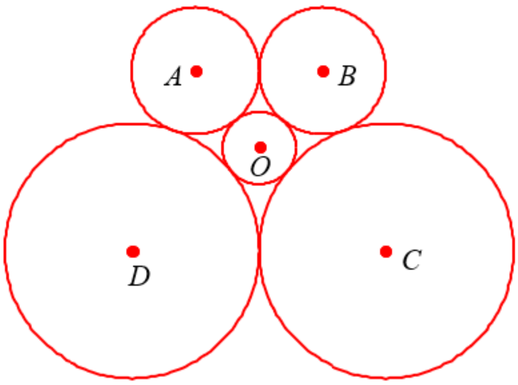
$$x^2 + y^2 = 25$$
$$x^2 + z^2 = 52$$
$$y^2 + z^2 = 45$$
$$x^2 + y^2 + z^2 = 61$$
$$x^2 = 16,\ y^2 = 9,\ z^2 = 36$$

Hence, $x = 4$, $y = 3$, $z = 6$, and the volume of the box is $8xyz = 576$.

Question S-29 [8 points]

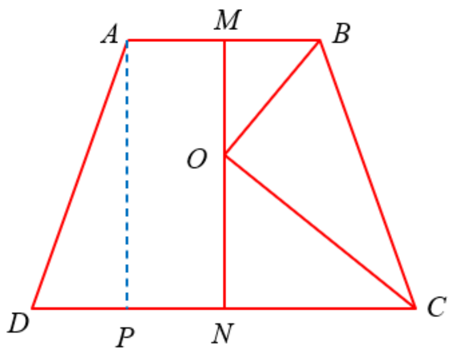
In the figure shown below, each of the circles with centers at A, B, C and D are tangent to two other circles and the circle with center at O. Given that the radii of the circles with centers A, B, C, D and O are R_1, R_1, R_2, R_2 and R respectively, and $R_2 = 2R_1$, $R = xR_1$, find the value of $462x$.

下图中，以 A，B，C 及 D 为圆心的四个圆分别与另外两个圆，以及以 O 为圆心的圆外切。已知以 A，B，C，D 及 O 为圆心的圆的半径分别为 R_1，R_1，R_2，R_2 及 R，且 $R_2 = 2R_1$，$R = xR_1$，求 $462x$ 的值。



Answer: [264]

Solutions:



Let M and N be the midpoints of AB and CD respectively, and let P be the foot of perpendicular from the point A to CD. Then $AM = MB = R_1$, $CN = ND = R_2 = 2R_1$.

$$AD = R_1 + R_2 = 3R_1$$

$$DP = R_2 - R_1 = R_1$$

$$AP = \sqrt{AD^2 - DP^2} = 2\sqrt{2}R_1$$

$$BO = R_1 + R = (1 + x)R_1$$

$$CO = R_2 + R = (2 + x)R_1$$

$$OM = \sqrt{OB^2 - MB^2} = R_1\sqrt{x^2 + 2x}$$

$$ON = \sqrt{OC^2 - NC^2} = R_1\sqrt{x^2 + 4x}$$

$$OM + ON = AP$$

$$\sqrt{x^2 + 2x} + \sqrt{x^2 + 4x} = 2\sqrt{2}$$

$$x^2 + 4x = x^2 + 2x - 4\sqrt{2}\sqrt{x^2 + 2x} + 8$$

$$2\sqrt{2x^2 + 4x} = 4 - x$$

$$4(2x^2 + 4x) = 16 - 8x + x^2$$

$$7x^2 + 24x - 16 = 0$$

$$(7x - 4)(x + 4) = 0$$

$$x = \frac{4}{7}$$

$$462x = 264$$

Question S-30 [8 points]

Given that $x_1, x_2, \ldots, x_{996}$ are real numbers with $-1 \leq x_i \leq 1$ for all i, and

$$x_1 + x_2 + \ldots + x_{996} = 0.$$

Find the maximum value of $\sum_{i=1}^{996} x_i^3$.

已知x_1，x_2，$\ldots$，x_{996}是实数。对于所有的i，$-1 \leq x_i \leq 1$，且

$$x_1 + x_2 + \ldots + x_{996} = 0.$$

求 $\sum_{i=1}^{996} x_i^3$ 的最大值。

Answer: [249]

Solutions:

Assume that k of the x_i's are positive, and l of them are negative, where $k + l = N \leq 996$. The remaining x_i's are zero.

Let $u_1, \ldots, u_k$ be those x_i's that are positive, and $v_1, \ldots, v_l$ are the negatives of those x_i's that are negative, so that $v_1, \ldots, v_l$ are positive.

Then we have

$$u_1 + \ldots + u_k = v_1 + \ldots + v_l.$$

and

$$\sum_{i=1}^{996} x_i^3 = (u_1^3 + \ldots + u_k^3) - (v_1^3 + \ldots + v_l^3).$$

Let $m = u_1 + \ldots + u_k = v_1 + \ldots + v_l$. Then $m \leq \min\{k, l\}$. By Holder's inequality,

$$\left(v_1^3 + \ldots + v_l^3\right)^{\frac{1}{3}} (1 + \ldots + 1)^{\frac{2}{3}} \geq v_1 + \ldots + v_l$$

with equality if and only if $v_1 = \ldots = v_l$. Therefore,

$$v_1^3 + \ldots + v_l^3 \geq \frac{m^3}{l^2}.$$

On the other hand, since $u_i \leq 1$, we find that $u_i^3 \leq u_i$.

Hence,

$$(u_1^3 + \ldots + u_k^3) - (v_1^3 + \ldots + v_l^3) \leq u_1 + \ldots + u_k - \frac{m^3}{l^2} = m - \frac{m^3}{l^2}.$$

With fixed l, set $f(m) = m - \dfrac{m^3}{l^2}$, $0 < m \leq \min\{l, N-l\}$.
Then $f'(m) = 1 - \dfrac{3m^2}{l^2}$.
If $k \geq \dfrac{l}{\sqrt{3}}$, then $f'(m) > 0$ when $0 < m < \dfrac{l}{\sqrt{3}}$ and $f'(m) < 0$ when $m > \dfrac{l}{\sqrt{3}}$.
This implies that $f(m)$ has a maximum value at $m = \dfrac{l}{\sqrt{3}}$, and the maximum value is

$$M_1 = \frac{l}{\sqrt{3}} - \frac{l}{3\sqrt{3}} = \frac{2}{3\sqrt{3}}l.$$

Notice that $N = k + l \geq \dfrac{l}{\sqrt{3}} + l = \dfrac{l}{\sqrt{3}}(\sqrt{3}+1)$.
Therefore,

$$M_1 \leq \frac{2}{3}\frac{N}{\sqrt{3}+1} < \frac{N}{4}.$$

If $k \leq \dfrac{l}{\sqrt{3}}$, then $f'(m) > 0$ when $0 < m < k$.
This implies that $f(m)$ has a maximum value at $m = k$, and the maximum value is

$$M_2 = k - \frac{k^3}{(N-k)^2}.$$

Fixed N and consider the function $g(k) = k - \dfrac{k^3}{(N-k)^2}$.
Then

$$g'(k) = 1 - \frac{3k^2}{(N-k)^2} - \frac{2k^3}{(N-k)^3} = 1 - 3z^2 - 2z^3 = -(z+1)^2(2z-1),$$

where $z = \dfrac{k}{N-k}$.
Hence, if $0 < z < \dfrac{1}{2}$, $g'(k) > 0$, and when $z > \dfrac{1}{2}$, $g'(z) < 0$.
This implies that $g(k)$ has a maximum value when $z = \dfrac{1}{2}$, or equivalently, when $k = \dfrac{N}{3}$. In this case, $l = \dfrac{2N}{3}$ and $k \leq \dfrac{l}{\sqrt{3}}$. Hence,

$$M_2 = k - \frac{k^3}{(N-k)^2} \leq \frac{N}{4}.$$

The whole analysis shows that the value of $\sum_{i=1}^{996} x_i^3$ cannot exceed $\dfrac{996}{4} = 249$.

When $x_1 = \ldots = x_{332} = 1$, $x_{333} = \ldots = x_{996} = -\dfrac{1}{2}$, we find that $\sum_{i=1}^{996} x_i^3 = 249$.

Hence, the maximum value of $\sum_{i=1}^{996} x_i^3$ is 249.