厦门大学马来西亚分校
陈景润杯中学数学竞赛

CHEN JINGRUN'S CUP SECONDARY SCHOOL MATHEMATICS COMPETITION

SENIOR CATEGORY

** 高阶组 **

Date: 1st April 2021   Time: 10:00 a.m. to 12:30 p.m.

日期：2021 年 4 月 1 日   时间：上午 10 时至下午 12 时 30 分

考生须知

Information and Instruction

这是**高阶组**的题目的线下版本。

This is the off-line version of the question paper for the **SENIOR** category.

负责老师会发给你一个 Microsoft Form 的链接，你必须上线在 Form 上填写答案提交。请务必在正确的 Form 上提交答案。

You need to go online to enter the answers to a Microsoft Form whose link is shared to you by your Teacher in Charge. Make sure you submit the correct form.

Microsoft Form 将在下午 12 点 30 分关闭，请在 12 点 25 分之前就按下"提交"键，以免因为网络问题而无法提交。

The Microsoft Form would be closed at 12:30pm. Please press the "Submit" button before 12:25pm, to avoid failure of submission due to network problems.

请仔细阅读下一页的资讯与指示，再继续作答。

Read through the information and instructions given in next page carefully. Then proceed to answer the questions.

在 Microsoft Form 的最后一页，你必须填写个人资料，再提交表格。请务必填写正确的**考生编号**。考生编号不是学号，也不是身份证号码。它是主办方提供的编号。如果你不清楚自己的考生编号，请询问学校的负责老师。

You are required to enter your personal information in the last page of the Microsoft Form, before you submit the form. Please enter your **candidate ID** correctly. The candidate ID is not your student ID, and is also not your IC number. It is a unique code given by the organizer. If you do not know your candidate ID, please ask the teacher in charge of your school.

一共有 30 题，**每题的答案都是一个整数。**每题在一个页面。 请在 Microsoft Form 相应的空格内填入答案，**答案必须是一个数字，不用写单位，不要填入数字以外的字元。**

There are altogether 30 questions. **The answer to each question is an integer.** Each question is on a single page. Please enter your answer in the corresponding empty box in the Microsoft Form. **The answer must be a number. Do not write the units. Do not enter characters other than numbers.**

第 1 至 第 20 题，每题 5 分	Question 1 to Question 20, each question carries 5 points
第 21 至 第 25 题，每题 6 分	Question 21 to Question 25, each question carries 6 points
第 26 至 第 30 题，每题 8 分	Question 26 to Question 30, each question carries 8 points
总分是 170 分	Total is 170 points.

学术诚信

Academic Honesty

参赛者必须独立参与比赛。不可得到其他人的协助，在比赛期间不能以任何方式和任何人互通信息。如果违反这些规则，参赛资格将被取消，而且也会受到学校的严厉纪律处分。

Participants should work independently in this competition. You should not get help from others, and should not communicate with anyone by any means during the competition. If you are found to violate these rules, you would be disqualified and face severe disciplinary action from your school.

关于线上提交答案的温馨提示

Kind Reminder for Submitting Answers to Microsoft Form online

1. "Submit"键在最后一页，在比赛时间结束前，请记得按下该键。
   The "Submit" button is at the last page. Before the end of the exam time, please remember to press the "Submit" button.

2. 有打 " * " 的项目一定要作答，否则无法提交。
   It is compulsory to response to all items with " * ", or else you would not be able to submit.

3. 未按“Submit" 键之前，一旦刷新或关掉浏览器，所有填入的资料将消失。
   If you refresh or close your browser before you press the "Submit" button, all the information that you have filled in will be lost.

4. 请在草稿纸上记录各题的答案，一旦不小心刷新或关掉浏览器，请重新登入 Microsoft Form 再填写一次。
   Please record your answers to each question on a piece of draft paper. If you accidentally refresh or close your browser, login the Microsoft form again to fill in all the answers.

5. 如果重复提交答案，必须提供合理的解释。
   If the form is submitted more than once, you need to give a good reason.

Question S-01 [5 points]

Given that the function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(3x + 2) = 2x + 2021$, find $f(2021)$.

已知函数 $f : \mathbb{R} \to \mathbb{R}$ 满足 $f(3x + 2) = 2x + 2021$，求 $f(2021)$。

Question S-02 [5 points]

In a game, each of the eight participants is given a card to write her own name on it. They then give the cards to the host. After mixing up the cards, the host randomly distributes the cards to the participants, each participant gets one. If x is the probability that exactly one of the participants does not get back the card with her own name written on it, find the value of $8!\,x$.

在一个游戏中，八位参与者各在一张卡片上写上自己的名字。她们之后将卡片交给主持人。主持人弄乱了卡片后，再任意将卡片发给这八位参与者，每人一张。若 x 为恰有一位参与者没有拿到写上自己名字的卡片的概率，求 $8!\,x$ 的值。

Question S-03 [5 points]

Given that a, b, c are three positive numbers with $(a+b-c):(b+c-a):(c+a-b)=6:7:9$ and $a+b+c=176$. Find b .

已知a, b, c 三个正数满足 $(a+b-c):(b+c-a):(c+a-b)=6:7:9$，且 $a+b+c=176$。求b。

Question S-04 [5 points]

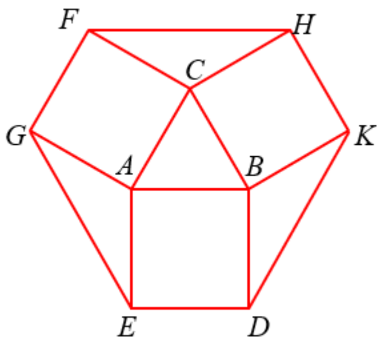
Let x be the number of ways to permute the letters in the word INSPECTION such that the two N's are not adjacent, find $\frac{x}{720}$.

设 x 为将INSPECTION 一字中的英文字母排列，两个N 不相邻的排列数。求 $\frac{x}{720}$。

Question S-05 [5 points]

In the figure below, ABC is an equilateral triangle. $ABDE$, $ACFG$ and $BCHK$ are squares. Given that $AB = 2$ and x is the area of the hexagon $DEGFHK$. Find the largest integer less than or equal to x.

下图中，ABC 是一个等边三角形。$ABDE$，$ACFG$ 及 $BCHK$ 是正方形。已知 $AB = 2$ 且 x 是六边形 $DEGFHK$ 的面积。求小于或等于 x 的最大整数。



Question S-06 [5 points]

Given that the set R consists of all pairs of real numbers (x, y) satisfying

$$\begin{aligned} 2x + y - 2 &\geq 0 \\ 2x - 3y + 6 &\geq 0 \\ x - 2y - 1 &\leq 0 \\ x + y - 7 &\leq 0 \end{aligned}$$

Find the maximum value of $x + 3y$ on R.

已知集合 R 的元素是所有满足不等式组

$$\begin{aligned} 2x + y - 2 &\geq 0 \\ 2x - 3y + 6 &\geq 0 \\ x - 2y - 1 &\leq 0 \\ x + y - 7 &\leq 0 \end{aligned}$$

的实数对 (x, y)。求 $x + 3y$ 在 R上的最大值。

Question S-07 [5 points]

Given that $\log_{10} 2 = 0.3010$, how many digits does the number 2^{2021} have?

已知 $\log_{10} 2 = 0.3010$，2^{2021} 这个数有几位数字？

Question S-08 [5 points]

Find the maximum value of the function $f(x) = \dfrac{5x^2 + 10x + 71}{x^2 + 2x + 7}$.

求函数 $f(x) = \dfrac{5x^2 + 10x + 71}{x^2 + 2x + 7}$ 的最大值。

Question S-09 [5 points]

Four people, P, Q, R and S, are accused in a trial. It is known that

- If P is guilty, then Q is guilty.
- If Q is guilty, then either R is guilty or P is not guilty.
- If S is guilty, then P is guilty and R is not guilty.
- If S is not guilty, then P is guilty.

Among these four accused P, Q, R, S, how many of them are guilty?

在一次审讯中有四位被告P，Q，R及S。已知

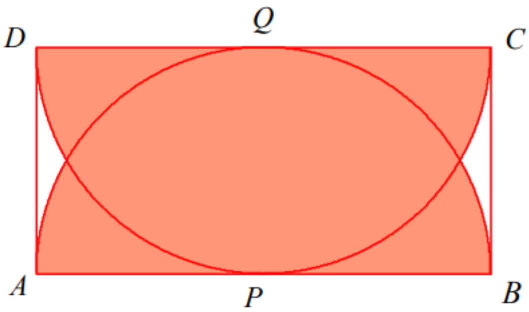
- 如果P是有罪的，则Q也是有罪的。
- 如果Q是有罪的，则R是有罪的或P是无罪的。
- 如果S是有罪的，则P是有罪的且R是无罪的。
- 若果S是无罪的，则P是有罪的。

P，Q，R，S四位被告中，有多少位有罪？

Question S-10 [5 points]

In the figure below, $ABCD$ is a rectangle with $AB = 12$, $AD = 6$. The arcs AQB and CPD are semicircles with diameters AB and CD respectively. If the area of the shaded region is x, find the integer that is closest to x.

下图中，$ABCD$ 是一个长方形，$AB = 12$，$AD = 6$。弧 AQB 及弧 CPD 为半圆，直径分别为 AB 及 CD。若阴影部分的面积为 x，求最靠近 x 的整数。



Question S-11 [5 points]

Given that $\log_a b = 317$, find the value of $\log_{\sqrt{a}}\left(\sqrt{a^5 b^3}\right)$.

已知 $\log_a b = 317$，求 $\log_{\sqrt{a}}\left(\sqrt{a^5 b^3}\right)$ 的值。

Question S-12 [5 points]

Find the sum of the integers n that satisfy the inequality $12n^2 < 383n - 2020$.

求满足不等式 $12n^2 < 383n - 2020$ 的整数 n 之和。

Question S-13 [5 points]

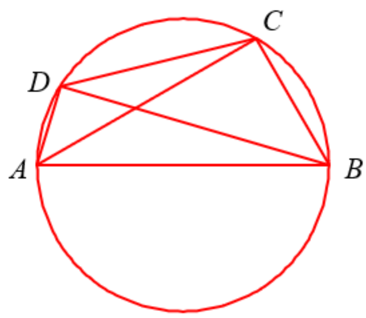
If a and b are two distinct positive numbers such that $11a - \frac{7}{b} = 11b - \frac{7}{a}$, find the value of $1001ab$.

若 a 及 b 是相异的正数且 $11a - \frac{7}{b} = 11b - \frac{7}{a}$，求 $1001ab$ 的值。

Question S-14 [5 points]

In the figure shown below, $ABCD$ is a cyclic quadrilateral, AB is a diameter of the circle. Given that $AD = 12$, $BC = 19$, $AC = x$, $BD = y$, find the value of $y^2 - x^2$.

下图中，$ABCD$ 是一圆内接四边形，AB 是圆的直径。已知 $AD = 12$，$BC = 19$，$AC = x$，$BD = y$，求 $y^2 - x^2$ 的值。



Question S-15 [5 points]

If n is a positive integer such that

$$2021\left[\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\ldots+\frac{1}{(2n-1)(2n+1)}\right]$$

is also an integer, find the largest possible value of n.

若 n 是一正整数使得

$$2021\left[\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\ldots+\frac{1}{(2n-1)(2n+1)}\right]$$

也是整数，求 n 的最大可能值。

Question S-16 [5 points]

If k is an integer such that $(k-8)x^2-(2k-5)x+(k-9)$ is negative for all real numbers x, find the largest possible value of k.

若 k 是一整数且对于所有的实数 x，$(k-8)x^2-(2k-5)x+(k-9)$ 的值都是负的，求 k 的最大可能值。

Question S-17 [5 points]

Let a, b, c be the three roots of the equation $x^3 - 99x - 101 = 0$. Find the value of $a^3 + b^3 + c^3$.

设 a，b，c 为方程式 $x^3 - 99x - 101 = 0$ 的三个根。求 $a^3 + b^3 + c^3$ 的值。

Question S-18 [5 points]

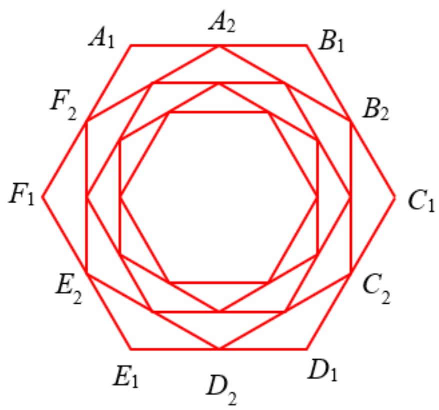
Under a rotation with respect to the point $P(a,b)$, the image of the point $A(11,5)$ is $A'(9,11)$. Find the value of $3b-a$.

已知经过以点 $P(a,b)$ 为中心的旋转后，点 $A(11,5)$ 的像是 $A'(9,11)$，求 $3b-a$ 的值。

Question S-19 [5 points]

In the figure below, $A_1B_1C_1D_1E_1F_1$ is a regular hexagon. The midpoints of the sides of $A_1B_1C_1D_1E_1F_1$ are joined to form the hexagon $A_2B_2C_2D_2E_2F_2$. The midpoints of the sides of $A_2B_2C_2D_2E_2F_2$ are joined to form the hexagon $A_3B_3C_3D_3E_3F_3$. This process continues to form the hexagons $A_nB_nC_nD_nE_nF_n$ for all $n \geq 2$. If the area of the hexagon $A_nB_nC_nD_nE_nF_n$ is S_n, and $S_1 = 228$, find the sum of the infinite series $S_1 + S_2 + S_3 + \ldots$.

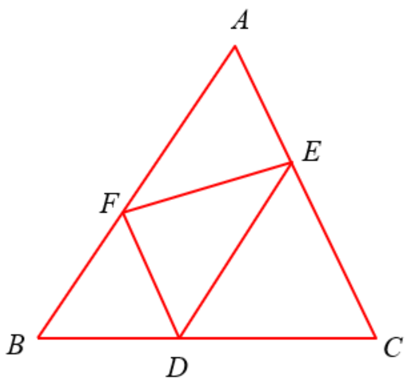
如下图所示，$A_1B_1C_1D_1E_1F_1$ 是一个正六边形。将 $A_1B_1C_1D_1E_1F_1$ 各边的中点连起来就得到六边形 $A_2B_2C_2D_2E_2F_2$。将 $A_2B_2C_2D_2E_2F_2$ 各边的中点连起来就得到六边形 $A_3B_3C_3D_3E_3F_3$。不断重复这过程以得到所有 $n \geq 2$ 时的六边形 $A_nB_nC_nD_nE_nF_n$。若六边形 $A_nB_nC_nD_nE_nF_n$ 的面积是 S_n，且 $S_1 = 228$，求无穷级数 $S_1 + S_2 + S_3 + \ldots$ 的和。



Question S-20 [5 points]

In the figure shown below, $AF : FB = 4 : 3$, $AE : EC = 3 : 5$, $BD : DC = 4 : 5$. Given that the areas of ΔABC and ΔDEF are S_1 and S_2 respectively. If $S_1 : S_2 = m : n$, where m and n are relatively prime positive integers, find the value of $m + n$.

下图中，$AF : FB = 4 : 3$，$AE : EC = 3 : 5$，$BD : DC = 4 : 5$。已知ΔABC及ΔDEF的面积分别为S_1及S_2。若m与n是互质的正整数使得$S_1 : S_2 = m : n$，求$m + n$的值。



Question S-21 [6 points]

Find the area of the region in the plane defined by the inequality $|x + 2y| + |x - 2y| \leq 34$.

求平面上由不等式 $|x + 2y| + |x - 2y| \leq 34$ 所定义的区域的面积。

Question S-22 [6 points]

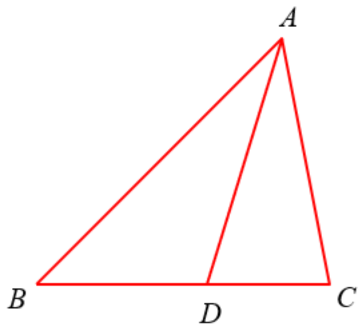
If $S=\sum_{k=1}^{112}\frac{(-1)^{k+1}}{\sqrt{4k^2-1}\left(\sqrt{2k+1}-\sqrt{2k-1}\right)}$, find the value of $120S$.

若 $S=\sum_{k=1}^{112}\frac{(-1)^{k+1}}{\sqrt{4k^2-1}\left(\sqrt{2k+1}-\sqrt{2k-1}\right)}$，求 $120S$ 的值。

Question S-23 [6 points]

In the figure below, AD bisects $\angle BAC$. Given that $AB = 9$, $BC = 10$, $AC = 6$. If $AD = x$, find the value of x^2.

下图中，AD 平分 $\angle BAC$。已知 $AB = 9$，$BC = 10$，$AC = 6$。若 $AD = x$，求 x^2 的值。



Question S-24 [6 points]

Find the largest integer less than 1000 which has exactly 7 positive factors.

求小于 1000 且恰有 7 个正因子的最大整数。

Question S-25 [6 points]

Given that $a_0 = 1$, $a_1 = \dfrac{1}{2}$, and for all $n \geq 1$, $a_{n+1} = \dfrac{a_{n-1}}{1 + n a_{n-1} a_n}$. Find $\dfrac{1}{a_{99} a_{100}}$.

已知 $a_0 = 1$，$a_1 = \dfrac{1}{2}$，且对于所有的 $n \geq 1$，$a_{n+1} = \dfrac{a_{n-1}}{1 + n a_{n-1} a_n}$，求 $\dfrac{1}{a_{99} a_{100}}$。

Question S-26 [8 points]

Consider the 2021 fractions

$$\frac{1}{2021}, \frac{2}{2021}, \frac{3}{2021}, \dots, \frac{2021}{2021}.$$

Let S be the set of these fractions which cannot be reduced to fractions with smaller denominators. For example, $\frac{3}{2021}$ is in S but $\frac{43}{2021}$ is not in S since $\frac{43}{2021} = \frac{1}{47}$. Find the sum of the elements in S.

考虑以下 2021 个分数：

$$\frac{1}{2021}, \frac{2}{2021}, \frac{3}{2021}, \dots, \frac{2021}{2021}.$$

设集合 S 是这些分数中不能被约简成分母比较小的分数所组成的集合。例如，$\frac{3}{2021}$ 在 S 中，而 $\frac{43}{2021}$ 不在 S 中，因为 $\frac{43}{2021} = \frac{1}{47}$。求 S 中的元素之和。

Question S-27 [8 points]

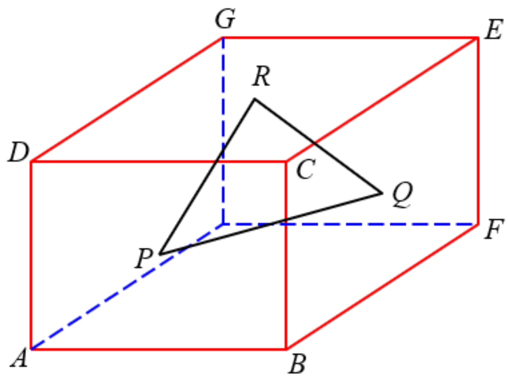
How many pairs of positive integers (m, n) satisfy the equation $\dfrac{1}{m} + \dfrac{1}{n} = \dfrac{3}{10000}$?

有多少对正整数 (m, n) 满足方程式 $\dfrac{1}{m} + \dfrac{1}{n} = \dfrac{3}{10000}$?

Question S-28 [8 points]

The figure below shows a rectangular box. Given that the points P, Q, R are the centers of the faces $ABCD$, $BCEF$ and $CDGE$ respectively. If the lengths of PQ, QR and PR are 5, $\sqrt{52}$ and $\sqrt{45}$ respectively, find the volume of the rectangular box.

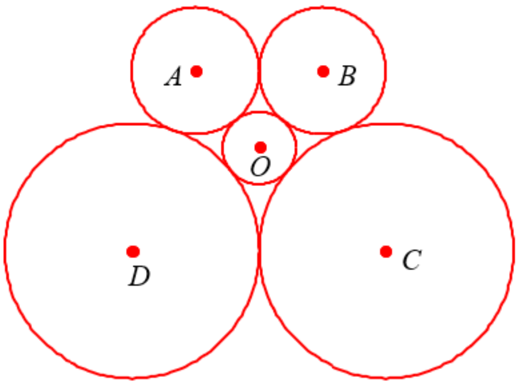
下图所示是一个长方体。已知P，Q，R分别是平面$ABCD$，$BCEF$及$CDGE$的中心。若PQ，QR 及PR的长分别为5，$\sqrt{52}$ 及$\sqrt{45}$，求长方体的体积。



Question S-29 [8 points]

In the figure shown below, each of the circles with centers at A, B, C and D are tangent to two other circles and the circle with center at O. Given that the radii of the circles with centers A, B, C, D and O are R_1, R_1, R_2, R_2 and R respectively, and $R_2 = 2R_1$, $R = xR_1$, find the value of $462x$.

下图中，以 A，B，C 及 D 为圆心的四个圆分别与另外两个圆，以及以 O 为圆心的圆外切。已知以 A，B，C，D 及 O 为圆心的圆的半径分别为 R_1，R_1，R_2，R_2 及 R，且 $R_2 = 2R_1$，$R = xR_1$，求 $462x$ 的值。



Question S-30 [8 points]

Given that $x_1, x_2, \ldots, x_{996}$ are real numbers with $-1 \leq x_i \leq 1$ for all i, and

$$x_1 + x_2 + \ldots + x_{996} = 0.$$

Find the maximum value of $\sum_{i=1}^{996} x_i^3$.

已知x_1，x_2，…，x_{996}是实数。对于所有的i，$-1 \leq x_i \leq 1$，且

$$x_1 + x_2 + \ldots + x_{996} = 0.$$

求 $\sum_{i=1}^{996} x_i^3$ 的最大值。